

# Homework Set 6 — Density matrix

Due June 20, 2021

1. The trace of an operator is defined as  $\text{Tr}\{A\} = \sum_m \langle m|A|m\rangle$ , where  $\{|m\rangle\}$  is an arbitrary orthonormal basis set. Introduce a second basis set, and use it to prove that the trace is independent of the choice of basis.

2. Prove the linearity of the trace operation by proving

$$\text{Tr}\{aA + bB\} = a\text{Tr}\{A\} + b\text{Tr}\{B\} \quad (1)$$

3. Prove the cyclic property of the trace by proving

$$\text{Tr}\{ABC\} = \text{Tr}\{BCA\} = \text{Tr}\{CAB\} \quad (2)$$

4. Which of the following density matrices correspond to a pure state?

$$\rho_1 = \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{5}{7} \end{pmatrix} \quad (3)$$

$$\rho_2 = \begin{pmatrix} \frac{1}{4} & i\frac{\sqrt{3}}{4} \\ -i\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix} \quad (4)$$

$$\rho_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (5)$$

$$\rho_4 = \begin{pmatrix} \frac{1}{5} & \frac{\sqrt{2}}{5} \\ \frac{\sqrt{2}}{5} & \frac{4}{5} \end{pmatrix} \quad (6)$$

5. The density matrix evolves with time. Derive the equation of motion

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] \quad (7)$$

using the Schrodinger equation and the most general form of the density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (8)$$

6. A spin- $\frac{1}{2}$  particle is in a statistical ensemble with a 50% probability to be in the  $|+_z\rangle$  state (the eigenstate of  $S_z$  with eigenvalue  $\hbar/2$ ) a 50% chance to be in  $|+_x\rangle$  (the eigenstate of  $S_x$  with eigenvalue  $\hbar/2$ ). [Note that these states are not orthogonal. Don't worry about that yet.] Use the standard procedure,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (9)$$

to write this density operator in terms of states in the  $|\pm_z\rangle$  basis, and then as a matrix using this basis. Use the density matrix to compute the probability that a measurement of the  $z$ -component of spin will return value  $+\hbar/2$ .

Now solve the eigenvalue/eigenvector problem for the density matrix. The eigenstates you find are eigenstates of spin along a definite axis — that is, eigenstates of  $S \cdot \hat{n}$  for some unit vector  $\hat{n}$ . Find  $\hat{n}$ . What is the entropy of the density matrix, and is it the same as you would guess at the beginning, knowing that there is a 50% chance to be in each of two states? What would be a more proper description of the density matrix, in terms of probabilities to be in orthogonal states?