

# Solutions:

## Homework Set 5 — Scattering 2

Due May 26, 2021

1. Without assuming a local potential, the general form of the Lippmann-Schwinger equation is

$$\Psi^+(\mathbf{r}) = \phi(\mathbf{r}) - \frac{2m}{\hbar^2} \int d^3r' d^3r'' G_{k'}^+(\mathbf{r}, \mathbf{r}') \langle \mathbf{r}' | V | \mathbf{r}'' \rangle \langle \mathbf{r}'' | \Psi^+ \rangle \quad (1)$$

$$= \phi(\mathbf{r}) - \frac{2m}{\hbar^2} \int d^3r' d^3r'' \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \lambda v(r') v(\mathbf{r}'') \Psi^+(\mathbf{r}'') \quad (2)$$

$$= \phi(\mathbf{r}) - \frac{\lambda}{4\pi} \frac{2m}{\hbar^2} \int d^3r' \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} v(r') \int d^3r'' v(\mathbf{r}'') \Psi^+(\mathbf{r}'') \quad (3)$$

We obtain the Born series by iteration

$$\begin{aligned} \Psi^+ = \phi(\mathbf{r}) - \frac{2m\lambda}{\hbar^2} \int d^3r' G_{k'}^+(\mathbf{r}, \mathbf{r}') v(r') \int d^3r'' v(\mathbf{r}'') \left[ \phi(\mathbf{r}'') \right. \\ \left. - \frac{2m\lambda}{\hbar^2} \int d^3r''' G_{k'}^+(\mathbf{r}', \mathbf{r}'') v(r'') \int d^3r'''' v(\mathbf{r}''') \Psi^+(\mathbf{r}''') \right] \end{aligned} \quad (4)$$

At every iteration, the common factors that appear are:

$$B \equiv \int d^3r' v(\mathbf{r}') \phi(\mathbf{r}') = \text{(independent of } \mathbf{r}) \quad (5)$$

$$A_{\mathbf{k}'}(\mathbf{r}') \equiv \frac{2m}{\hbar^2} \int d^3r'' G_{k'}^+(\mathbf{r}', \mathbf{r}'') v(\mathbf{r}'') \quad (6)$$

So we can rewrite Eq. (4) as

$$\Psi^+(\mathbf{r}) = \phi(\mathbf{r}) - \lambda A_{\mathbf{k}'}(\mathbf{r}) \left[ B - \lambda \int d^3r'' A_{\mathbf{k}'}(\mathbf{r}'') \left( B - \lambda \int d^3r''' A_{\mathbf{k}'}(\mathbf{r}''') \left\{ \dots \right\} \right) \right] \quad (7)$$

$$= \phi(\mathbf{r}) - \lambda A_{\mathbf{k}'}(\mathbf{r}) B + \lambda^2 A_{\mathbf{k}'}(\mathbf{r}) \int d^3r'' A_{\mathbf{k}'}(\mathbf{r}'') B - \lambda^3 \dots \quad (8)$$

The first Born approximation is then:

$$\Psi^+(\mathbf{r}) = \phi(\mathbf{r}) - \lambda A_{\mathbf{k}'}(\mathbf{r})B \quad (9)$$

$$= \phi(\mathbf{r}) - \lambda \frac{2m}{\hbar^2} \int d^3r' \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} v(\mathbf{r}') \int d^3r'' v(\mathbf{r}'') e^{i\mathbf{k}\cdot\mathbf{r}''} \quad (10)$$

$$\xrightarrow{r \rightarrow \infty} \phi(\mathbf{r}) - \frac{e^{ikr}}{r} \lambda \frac{m}{2\pi\hbar^2} \int d^3r' e^{i\mathbf{k}'\cdot\mathbf{r}'} v(\mathbf{r}') \int d^3r'' e^{i\mathbf{k}\cdot\mathbf{r}''} v(\mathbf{r}'') \quad (11)$$

$$\implies f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{\lambda m}{2\pi\hbar^2} F_{\mathbf{k}}(v) F_{\mathbf{k}'}(v) \quad (12)$$

where  $F$  is the Fourier transform of  $v$ .

$$F_{\mathbf{k}}(v) = \int d^3r e^{i\mathbf{k}\cdot\mathbf{r}} v(\mathbf{r}). \quad (13)$$

This is in contrast to the case of separable potential, which is proportional to the Fourier transform with respect to the momentum transfer  $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$ :  $F_{\mathbf{q}}(V)$

2. (a) Let  $R(r)$  be the radial part of wave function, and

$$\chi(r) = rR(r) \quad (14)$$

with boundary condition

$$\chi(r \rightarrow 0) \rightarrow 0 \quad (15)$$

$\chi$  satisfies

$$\left( \frac{d^2}{dr^2} + k^2 - k_0^2 \right) \chi(r) = 0 \quad (16)$$

with

$$k_0^2 = \frac{2mV_0}{\hbar^2} \theta(a - r) \quad (17)$$

For  $r > a$ ,  $\chi$  is solution to free equation. The  $s$ -wave is a linear combination of  $\sin(kr)$  and  $\cos(kr)$ , or equivalently  $\sin(kr + \delta_0)$ .

Similarly, for  $r < a$ ,  $\chi$  is a linear combination of  $\sinh(\alpha r)$  and  $\cosh(\alpha r)$  with  $\alpha^2 = k_0^2 - k^2 > 0$  (note that  $E < V_0$ ).

The boundary condition allows only the cosh term.

$$\chi(r) = \begin{cases} A \sinh(\alpha r) & r < a \\ B \sin(kr + \delta_0) & r > a \end{cases} \quad (18)$$

At  $r = a$ , match  $\chi$  and  $\chi'$ :

$$A \sinh(\alpha a) = B \sin(ka + \delta_0) \quad (19)$$

$$\alpha A \cosh(\alpha a) = kB \cos(ka + \delta_0) \quad (20)$$

$$\implies k \tanh(\alpha a) = \alpha \tan(ka + \delta_0) \quad (21)$$

$$\delta_0(k) = -ka + \tan^{-1} \left[ \frac{k}{\sqrt{k_0^2 - k^2}} \tanh \left( a \sqrt{k_0^2 - k^2} \right) \right] \quad (22)$$

(b) For  $V \rightarrow \infty$ ,  $\tanh \rightarrow 1 \implies \delta_0 \rightarrow -ka$

(c)

$$\lim_{k \rightarrow 0} \delta = -k\gamma \quad (23)$$

$$\gamma = a - \frac{1}{k_0} \tanh(k_0 a) \quad (24)$$

$$\sigma = \lim_{k \rightarrow 0} \left[ \frac{4\pi \sin^2 \delta}{k^2} \right] \quad (25)$$

$$= 4\pi \lim_{k \rightarrow 0} \left[ \frac{\delta}{k} \right]^2 \quad (26)$$

$$= 4\pi \gamma^2 \quad (27)$$

$$= 4\pi \left[ a - \frac{1}{k_0} \tanh(k_0 a) \right]^2 \quad (28)$$

3. In the limit of low energy, only the  $s$ -wave contributes,  $l = 0$ :

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{d^2}{dr^2} r - \frac{l(l+1)}{r^2} \right) + V(r) - \frac{\hbar^2 k^2}{2m} \right] R_0(r) = 0 \quad (29)$$

Let  $\chi(r) \equiv rR(r)$ :

$$\left[ \frac{d^2}{dr^2} + \gamma \delta(r - a) + k^2 \right] \chi(r) = 0 \quad (30)$$

with  $\gamma \equiv \frac{2mC}{\hbar^2}$ . The general solution inside (regular at the origin) and outside the shell is:

$$\chi = \begin{cases} A \sin(kr) & r < a \\ B \sin(kr + \delta) & r > a \end{cases} \quad (31)$$

$\chi$  should be continuous at  $a$ , and  $\chi'$  should have the correct discontinuity for the  $\delta$  function:

$$A \sin(ka) = B \sin(ka + \delta) \quad (32)$$

$$0 = \frac{d\chi}{dr} \Big|_{a^+} - \frac{d\chi}{dr} \Big|_{a^-} + \gamma \chi(a) \quad (33)$$

$$= kB \cos(ka + \delta) - Ak \cos(ka) + \gamma A \sin(ka) \quad (34)$$

$$\implies \tan(ka + \delta) = \frac{\tan(ka)}{1 - \frac{\gamma}{k} \tan(ka)} \quad (35)$$

with  $\gamma = \frac{2mC}{\hbar^2}$ .

For low energy,  $k \rightarrow 0$ :

$$\delta = \tan^{-1} \left[ \frac{\tan(ka)}{1 - \frac{\gamma}{k} \tan(ka)} \right] - ka \quad (36)$$

$$\simeq \frac{a^2 \gamma k}{1 - \gamma a} + O(k^3) \quad (37)$$

$$\sigma \simeq \frac{4\pi}{k^2} \delta^2 \quad (38)$$

$$= \frac{4\pi \gamma^2 a^2}{(1 - \gamma a)^2} \quad (39)$$

4. (a) In the Born approximation:

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = \frac{-m}{2\pi\hbar^2} \int d^3x' e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}'} V(\mathbf{x}') \quad (40)$$

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\mathbf{k}', \mathbf{k})|^2 \quad (41)$$

$$= \frac{m^2}{4\pi^2\hbar^4} \int d^3x' d^3x'' e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}'} V(\mathbf{x}') e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}''} V(\mathbf{x}'') \quad (42)$$

$$= \frac{m^2}{4\pi^2\hbar^4} \int d^3x' d^3x'' V(\mathbf{x}') V(\mathbf{x}'') e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} e^{-i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x}'')} \quad (43)$$

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} \quad (44)$$

$$= \frac{m^2}{4\pi^2\hbar^4} \int d^3x' d^3x'' V(\mathbf{x}') V(\mathbf{x}'') e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} \int d\Omega_{\mathbf{k}'} e^{-i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x}'')} \quad (45)$$

$$= \frac{m^2}{4\pi^2\hbar^4} \int d^3x' d^3x'' V(\mathbf{x}') V(\mathbf{x}'') e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} 2\pi \int_{-1}^1 d\cos\theta e^{ik'|\mathbf{x}'-\mathbf{x}''|\cos\theta} \quad (46)$$

$$= \frac{m^2}{4\pi^2\hbar^4} \int d^3x' d^3x'' V(\mathbf{x}') V(\mathbf{x}'') e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} 4\pi \frac{\sin(k|\mathbf{x}'-\mathbf{x}''|)}{k|\mathbf{x}'-\mathbf{x}''|} \quad (47)$$

(Recall  $k' = k$  from energy conservation in elastic scattering).

In this case  $V(\mathbf{x}) = V(r)$ , and the cross section cannot depend on the direction of  $\mathbf{k}$  by symmetry  $\implies$  we can average the expression over angle and replace

$$e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} \rightarrow \langle e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} \rangle \quad (48)$$

$$= \frac{1}{2} \int_{-1}^1 d\cos\theta e^{ik|\mathbf{x}'-\mathbf{x}''|\cos\theta} \quad (49)$$

$$= \frac{\sin(k|\mathbf{x}'-\mathbf{x}''|)}{k|\mathbf{x}'-\mathbf{x}''|} \quad (50)$$

$$\implies \sigma = \frac{m^2}{\pi\hbar^4} \int d^3x' d^3x'' V(r') V(r'') \frac{\sin^2(k|\mathbf{x}'-\mathbf{x}''|)}{k^2|\mathbf{x}'-\mathbf{x}''|^2} \quad (51)$$

(b) The optical theorem in the second Born approximation reads:

$$\sigma = \frac{4\pi}{k} \Im [f^{(2)}(\mathbf{k} = \mathbf{k}')] \quad (52)$$

$$= \Im \left[ \frac{4\pi}{k} \left( -\frac{m}{2\pi\hbar^2} \right)^2 \int d^3x' d^3x'' e^{-i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} \frac{e^{ik|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}'-\mathbf{x}''|} V(r') V(r'') \right] \quad (53)$$

Again we can perform the angular integral explicitly

$$\sigma = \Im \left[ -\frac{m^2}{\pi \hbar^4 k} \int d^3 x' d^3 x'' \frac{\sin(k|\mathbf{x}' - \mathbf{x}''|)}{k|\mathbf{x}' - \mathbf{x}''|} \frac{e^{ik|\mathbf{x}' - \mathbf{x}''|}}{|\mathbf{x}' - \mathbf{x}''|} V(r') V(r'') \right] \quad (54)$$

$$= \frac{m^2}{\pi \hbar^4} \int d^3 x' d^3 x'' V(r') V(r'') \frac{\sin^2(k|\mathbf{x}' - \mathbf{x}''|)}{k^2 |\mathbf{x}' - \mathbf{x}''|^2} \quad (55)$$

5. (a) In the Born approximation:

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = \frac{-m}{2\pi \hbar^2} \int d^3 x' e^{-i\mathbf{q}\cdot\mathbf{r}'} V(\mathbf{r}') \quad (56)$$

$$= \frac{-mV_0}{2\pi \hbar^2} \int d^3 x' e^{-i\mathbf{q}\cdot\mathbf{r}'} [\delta^{(3)}(\mathbf{r} - \epsilon \hat{z}) - \delta^{(3)}(\mathbf{r} + \epsilon \hat{z})] \quad (57)$$

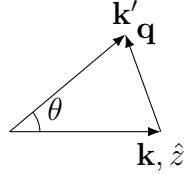
$$= \frac{-mV_0}{2\pi \hbar^2} (e^{i\epsilon\mathbf{q}\cdot\hat{z}} - e^{-i\epsilon\mathbf{q}\cdot\hat{z}}) \quad (58)$$

$$= \frac{-mV_0}{2\pi \hbar^2} i \sin(\epsilon q_z) \quad (59)$$

with

$$q_z \equiv (\mathbf{k}' - \mathbf{k})_z \quad (60)$$

$$= -2k \sin^2 \frac{\theta}{2} \quad (61)$$



$$\frac{d\sigma}{d\Omega} = |f|^2 \quad (62)$$

$$= \frac{m^2 V_0^2}{4\pi^2 \hbar^4} \sin^2(-2k\epsilon \sin^2 \frac{\theta}{2}) \quad (63)$$

(b) We expect the Born approximation to be good when

$$1 \gg \left| \frac{2m}{\hbar^2} \int d^3x' \frac{e^{ik|x'|}}{4\pi|x'|} V(x') e^{i\vec{k}\cdot\vec{x}'} \right| \quad (64)$$

$$= \left| \frac{2mV_0}{\hbar^2} \frac{e^{ik|\epsilon|}}{4\pi|\epsilon|} [e^{i\epsilon k_z} - e^{-i\epsilon k_z}] \right| \quad (65)$$

$$= \left| \frac{2mV_0}{\hbar^2} \frac{e^{ik|\epsilon|}}{4\pi|\epsilon|} 2i \sin(\epsilon k) \right| \quad (66)$$

$$= \left| \frac{mV_0}{\hbar^2} \frac{e^{ik|\epsilon|}}{\pi|\epsilon|} i \sin(\epsilon k) \right| \quad (67)$$

$$= \left| \frac{mV_0}{\hbar^2} \frac{e^{ik|\epsilon|+i\pi}}{\pi|\epsilon|} \sin(\epsilon k) \right| \quad (68)$$

$$= \left| \frac{mV_0}{\hbar^2 \pi \epsilon} \right| |e^{ik|\epsilon|+i\pi}| |\sin(\epsilon k)| \quad (69)$$

$$= \frac{m|V_0|}{\pi \hbar^2 \epsilon} |\sin(\epsilon k)| \quad (70)$$

$$\implies |V_0| \ll \frac{\pi \epsilon \hbar^2}{m |\sin(\epsilon k)|} \quad (71)$$

6. (a) The potential energy  $V(\mathbf{r})$  is related to the charge distribution by

$$\nabla^2 V(r) = -4\pi e \rho(r) \quad (72)$$

(in Gaussian units.) Taking the Fourier transform, this reads

$$\tilde{V}(q) = \frac{4\pi e}{q^2} \tilde{\rho}(q) \quad (73)$$

In the Born approximation, the scattering amplitude is proportional to the Fourier transform of  $V$

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = \frac{-m}{2\pi \hbar^2} \tilde{V}(q) \quad (74)$$

$$= \frac{-2me}{\hbar^2} \frac{\tilde{\rho}(q)}{q^2} \quad (75)$$

with  $q = |\mathbf{k}' - \mathbf{k}| = 2k \sin \frac{\theta}{2}$ .

$$\frac{d\sigma}{d\Omega} = |f|^2 \quad (76)$$

$$= \frac{4e^2 m^2 \tilde{\rho}^2(2k \sin \frac{\theta}{2})}{\hbar^4 (2k \sin \frac{\theta}{2})^4} \quad (77)$$

(b) For  $\theta \rightarrow 0$ ,  $q \rightarrow 0$ .

$$\frac{d\sigma}{d\Omega}(\theta = 0) = \lim_{q \rightarrow 0} \frac{4e^2 m^2 \tilde{\rho}^2(q)}{\hbar^4 q^4} \quad (78)$$

Expanding for small  $q$ :

$$\tilde{\rho}(q) = \int d^3x \rho(r) e^{i\mathbf{q} \cdot \mathbf{x}} \quad (79)$$

$$= \int d^3x \rho(r) \left[ 1 - i\mathbf{q} \cdot \mathbf{x} - \frac{(\mathbf{q} \cdot \mathbf{x})^2}{2} + \dots \right] \quad (80)$$

$$= \int d^3x \rho(r) - i \int d^3x q r \cos \theta \rho(r) - \frac{q^2}{2} \int d^3x r^2 \cos^2 \theta \rho(r) + \dots \quad (81)$$

where  $\theta$  is chosen to be the angle with respect to  $\mathbf{q}$ . The first term is zero because the net charge is zero, and the second term is zero from rotational symmetry.

We can perform the integral over  $\theta$ :

$$\int_{-1}^1 d(\cos \theta) \cos^2 \theta = \frac{2}{3} \quad (82)$$

$$= \int_{-1}^1 d(\cos \theta) \frac{1}{3} \quad (83)$$

$$\implies \lim_{q \rightarrow 0} \tilde{\rho}(q) = -\frac{q^2}{6} \int d^3x r^2 \rho(r) \quad (84)$$

$$= -\frac{q^2 A}{6} \quad (85)$$

$$\frac{d\sigma}{d\Omega}(\theta = 0) = \frac{e^2 m^2 A^2}{9\hbar^4} \quad (86)$$

(c) A hydrogen atom consists of a nucleus at the center, and an electron, whose charge



distribution is given by the square of the ground state wave function  $\Psi_{1s}^2(r)$ :

$$\rho(r) = -e [\delta^3(\mathbf{x}) - \Psi_{1s}^2(r)] \quad (87)$$

$$\Psi_{1s}(r) = \frac{2e^{-r/a_0}}{\sqrt{a_0^3}} \quad (88)$$

$$\tilde{\rho}(q) = -e \int d^3x \left[ \delta^3(\mathbf{x}) - \frac{4}{a_0^3} e^{-2r/a_0} \right] e^{i\mathbf{q}\cdot\mathbf{x}} \quad (89)$$

$$= -e \left[ 1 - \frac{16}{(4 + q^2 a_0^2)^2} \right] \quad (90)$$

$$= -e \frac{q^2 a_0^2}{2} + O(q^4) \quad (91)$$

$$A = -6 \lim_{q \rightarrow 0} \frac{\tilde{\rho}(q)}{q^2} \quad (92)$$

$$= 3ea_0^2 \quad (93)$$

$$\frac{d\sigma}{d\Omega}(\theta \rightarrow 0) = a_0^2 \quad (94)$$

7. Consider the (non-relativistic) scattering of an electron of mass  $m$  and momentum  $k$  through an angle  $\theta$ . Calculate the differential cross-section  $\frac{d\sigma}{d\Omega}$  in the Born approximation for the spin-dependent potential

$$V = e^{-\mu r^2} [A + B\vec{\sigma} \cdot \mathbf{r}], \quad (95)$$

where  $\vec{\sigma}$  are the Pauli matrices and  $\mu, A, B$  are constants. Assume that the initial spin is polarized along the incident direction and sum over all final spins.

The Born approximation has the form

$$f^{(1)} = \frac{m(2\pi)^2}{\hbar^2} \langle \mathbf{k}', m'_s | V | \mathbf{k}, m_s \rangle \quad (96)$$

The initial spin is polarized,  $m_s = +1/2$ , and we want the cross section summed over final spins:

$$\frac{d\sigma}{d\Omega} = \frac{m^2(2\pi)^4}{\hbar^4} [|\langle \mathbf{k}', + | V | \mathbf{k}, + \rangle|^2 + |\langle \mathbf{k}', - | V | \mathbf{k}, + \rangle|^2] \quad (97)$$

$$= \frac{m^2}{4\pi^2 \hbar^4} [|\langle + | \tilde{V}(q) | + \rangle|^2 + |\langle - | \tilde{V}(q) | + \rangle|^2] \quad (98)$$

The spatial part is proportional to the Fourier transform, as usual, and we are interested only in the spin transitions  $+ \rightarrow +$  and  $+ \rightarrow -$

$$\tilde{V}(\mathbf{q}) = \langle \mathbf{k}' | V | \mathbf{k} \rangle \quad (99)$$

$$= \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) \quad (100)$$

$$= \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} e^{-\mu r^2} [A + B\vec{\sigma} \cdot \mathbf{x}] \quad (101)$$

$$= e^{-q^2/4\mu} \int d^3x e^{-(\sqrt{\mu}\mathbf{x} - \frac{i}{2\sqrt{\mu}}\mathbf{q})^2} [A + B\vec{\sigma} \cdot \mathbf{x}] \quad (102)$$

$$= e^{-q^2/4\mu} \int d^3x e^{-(\sqrt{\mu}\mathbf{x} - \frac{i}{2\sqrt{\mu}}\mathbf{q})^2} \left[ A + \frac{1}{\sqrt{\mu}} B\vec{\sigma} \cdot \left( \sqrt{\mu}\mathbf{x} - \frac{i}{2\sqrt{\mu}}\mathbf{q} \right) + \frac{i}{2\mu} B\vec{\sigma} \cdot \mathbf{q} \right] \quad (103)$$

$$= \frac{1}{\mu^{3/2}} e^{-q^2/4\mu} \int d^3x' e^{-\mathbf{x}'^2} \left[ A + \frac{1}{\sqrt{\mu}} B\vec{\sigma} \cdot \mathbf{x}' + \frac{i}{2\mu} B\vec{\sigma} \cdot \mathbf{q} \right] \quad (104)$$

$$= \frac{1}{\mu^{3/2}} e^{-q^2/4\mu} \left[ A + \frac{i}{2\mu} B\mathbf{q} \cdot \vec{\sigma} \right] \int d^3x' e^{-\mathbf{x}'^2} + \frac{1}{\mu^{3/2}} e^{-q^2/4\mu} \frac{B}{\sqrt{\mu}} \vec{\sigma} \cdot \left[ \int d^3x' \mathbf{x}' e^{-\mathbf{x}'^2} \right] \quad (105)$$

$$= \frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \left[ A + \frac{i}{2\mu} B\mathbf{q} \cdot \vec{\sigma} \right] + 0 \quad (106)$$

$$= \frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \left[ A + \frac{i}{2\mu} B\mathbf{q} \cdot \vec{\sigma} \right] \quad (107)$$

(which is a 2x2 matrix)

Choose the incoming wave in the  $z$  direction, and write the transverse components in the basis

$$\sigma^\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y) \quad (108)$$

$$q^\pm = q_x \pm iq_y \quad (109)$$

$$\implies \tilde{V}(\mathbf{q}) = \frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \left[ A + \frac{i}{2\mu} B(q_z\sigma_z + q^+\sigma^- + q^-\sigma^+) \right] \quad (110)$$

The transition  $+\rightarrow+$  only has contribution from the diagonal terms ( $A$  and the term with  $\sigma_z$ ). The transition  $+\rightarrow-$  only has contribution from the lowering operator  $\sigma^-$ . The raising operator  $\sigma^+$  has no affect when the incoming spin is  $+$ :

$$\langle + | \sigma_z | + \rangle = 1 \quad (111)$$

$$\langle - | \sigma^- | + \rangle = 1 \quad (112)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left[ \frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \right]^2 \left[ \left| A + \frac{iq_z}{2\mu} B \right|^2 + \left| \frac{iq^+ B}{2\mu} \right|^2 \right] \quad (113)$$

$$= \frac{m^2}{4\pi^2\hbar^4} \left[ \frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \right]^2 \left[ A^2 + \frac{B^2 q^2}{4\mu^2} \right] \quad (114)$$

$$= \frac{m\pi}{4\hbar^4 \mu^3} e^{-q^2/2\mu} \left[ A^2 + \frac{B^2 q^2}{4\mu^2} \right] \quad (115)$$

$$= \frac{m\pi}{4\hbar^4 \mu^3} e^{-k^2(1-\cos\theta)/\mu} \left[ A^2 + \frac{B^2 k^2(1-\cos\theta)}{2\mu^2} \right] \quad (116)$$