Solutions:
Homework Set 4 — Scattering 1

Due May 12, 2021

1. At 100 m radius, 10 m^2 covers

$$\frac{4\pi 10\mathrm{m}^2}{4\pi(100\mathrm{m})^2} = 10^{-3} \text{ steradians} = \Omega. \tag{1}$$

$d\sigma/d\Omega$ is approximately constant over this solid angle (compared, e.g., to our one significant figure knowledge of the incoming flux),

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{d\sigma}{d\Omega}\Big|_{\theta=30^\circ} \int d\Omega \tag{2}$$

$$= 10^{-34} \sin^2(30^\circ)\mathrm{cm}^2 \cdot 10^{-3} \tag{3}$$

$$= \frac{1}{4} \times 10^{-37}\mathrm{cm}^2 = 2.5 \times 10^{-38}\mathrm{cm}^2 \tag{4}$$

The total number of particles is this times the incoming flux:

$$\frac{2 \times 10^{32} \cdot 2.5 \times 10^{-38} \text{ particles}}{\text{unit time}} = 5 \times 10^{-6} \text{ particles/time} \tag{5}$$

2. (a) The $E \to E + i\epsilon$ prescription is correct:

$$
\begin{aligned}
G_+(x,x') &= \frac{\hbar^2}{2m}\langle x|\frac{1}{E-H_0+i\epsilon}|x'\rangle & (6)\\
&= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}dp'\langle x|p\rangle\langle p|\frac{1}{E-H_0+i\epsilon}|p'\rangle\langle p'|x'\rangle & (7)\\
&= \frac{\hbar^2}{2m}\frac{1}{2\pi\hbar}\int dpdp' e^{ipx/\hbar}\frac{\delta(p-p')}{E-H_0+i\epsilon}e^{-ip'x'/\hbar} & (8)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}dk'\frac{e^{ik'(x-x')}}{k^2-k'^2+i\epsilon} \qquad (p'=\hbar k', E=\frac{\hbar^2k^2}{2m}) & (9)\\
&= -\frac{1}{2\pi}\int_{-\infty}^{\infty}dk'\frac{e^{ik'(x-x')}}{[k'+(k+i\epsilon)][k'-(k+i\epsilon)]} & (10)\\
&= -\frac{1}{2\pi}\begin{cases} 2\pi i\frac{1}{2k}e^{ik(x-x')}, & x>x' \text{ (close contour in upper half plane)}\\ -2\pi i\frac{1}{2k}e^{-ik(x-x')}, & x<x' \text{ (close contour in lower half plane)}\end{cases} & (11)\\
&= -\frac{i}{2k}e^{ik|x-x'|} & (12)\\
\langle x|\Psi^+\rangle &= \langle x|\phi\rangle - \frac{2m}{\hbar^2}\int_{-\infty}^{\infty}dx'\frac{i}{2k}e^{ik|x-x'|}V(x')\langle x'|\Psi^+\rangle \quad \text{(assuming local } V\text{)} & (13)
\end{aligned}
$$

(b)

$$V = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x), \quad (\gamma > 0). \tag{14}$$

$$\implies \langle x|\Psi^+\rangle = \langle x|\phi\rangle - \frac{2m}{\hbar^2}\int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|} V(x')\langle x'|\Psi^+\rangle \tag{15}$$

$$= \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{2m}{\hbar^2}\int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|}\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x')\langle x'|\Psi^+\rangle \tag{16}$$

$$= \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k} e^{ik|x|}\langle 0|\Psi^+\rangle \tag{17}$$

$$\langle 0|\Psi^+\rangle = \frac{1}{\sqrt{2\pi}} + \frac{i\gamma}{2k}\langle 0|\Psi^+\rangle \tag{18}$$

$$\implies \langle 0|\Psi^+\rangle = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{i\gamma}{2k}\right)^{-1} \tag{19}$$

$$\implies \langle x|\Psi^+\rangle = \frac{1}{\sqrt{2\pi}}\left(e^{ikx} + e^{ik|x|}\frac{i\gamma}{2k}\left(1 - \frac{i\gamma}{2k}\right)^{-1}\right) \tag{20}$$

$$= \frac{1}{\sqrt{2\pi}}\begin{cases} \left(1 - \frac{i\gamma}{2k}\right)^{-1} e^{ikx}, & x > 0 \\ e^{ikx} + \left(\frac{2k}{i\gamma} - 1\right)^{-1} e^{-ikx}, & x < 0 \end{cases} \tag{21}$$

$$\implies T = \left(1 - \frac{i\gamma}{2k}\right)^{-1},$$

$$R = \left(\frac{2k}{i\gamma} - 1\right)^{-1} \tag{22}$$

(c) Analytically continuing the wavefunction to complex k, the poles of T and R are both at $k = i\gamma/2$, and $\langle x|\Psi\rangle \sim e^{-\gamma|x|/2}$.

The Schrodinger equation,

$$-\frac{1}{2m}\frac{d^2\Psi}{dx^2} - \frac{\gamma}{2m}\delta(x)\Psi = -|E|\Psi, \tag{23}$$

has solutions of the form

$$\Psi \sim \begin{cases} e^{-\kappa x} & (x > 0) \\ e^{\kappa x} & (x < 0), \end{cases} \tag{24}$$

with $k = i\kappa = i\sqrt{2m|E|}$, satisfying

$$\left.\frac{d\Psi}{dx}\right|_{0^+} - \left.\frac{d\Psi}{dx}\right|_{0^-} = -\gamma\Psi(0) \tag{25}$$

which implies that $\kappa = \gamma/2$, or $k = i\gamma/2$.

Thus $\Psi \sim e^{-\gamma|x|/2}$, in agreement with the expectation that bound states are represented by poles in the scattering amplitude on the imaginary momentum axis.

3. (a)

$$E = \frac{\hbar^2 k^2}{2m} > V_0 \tag{26}$$

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & x < 0 \\ te^{ik'x} & x > 0 \end{cases} \tag{27}$$

$$k'^2 = k^2 - \frac{2mV_0}{\hbar^2} \tag{28}$$

Match at x=0:

$$\begin{cases} 1 + r = t \\ ik(1-r) = ik't \end{cases} \tag{29}$$

$$r = \frac{k-k'}{k+k'} \tag{30}$$

$$t = \frac{2k}{k+k'} \tag{31}$$

$$R = |r|^2 = \frac{(k-k')^2}{(k+k')^2} \tag{32}$$

$$T = |t|^2 = \frac{4k^2}{(k+k')^2} \tag{33}$$

(b)

$$J_R = \frac{\hbar k}{m} R = \frac{\hbar}{m} \frac{k(k-k')^2}{(k+k')^2} \tag{34}$$

$$J_T = \frac{\hbar k'}{m} T = \frac{\hbar}{m} \frac{4k'k^2}{(k+k')^2} \tag{35}$$

$$J_R + J_T = \frac{\hbar k}{m} \tag{36}$$

4. (a) The quantity to be calculated,

$$\frac{\hbar^2}{2m} \langle \mathbf{x} | \frac{1}{E - H_0 + i\epsilon} | \mathbf{x}' \rangle, \tag{37}$$

is the Green's function for (-) the Helmholtz operator $(\nabla^2 + k^2)$:

$$(\nabla^2 + k^2)\frac{\hbar^2}{2m}\langle \mathbf{x}|\frac{1}{E - H_0 + i\epsilon}|\mathbf{x}'\rangle = (\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x}, \mathbf{x}'). \tag{38}$$

$[\nabla^2 + k^2, L^2] = 0$, so they can be simultaneously diagonalized, and we look for a solution of the form:

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l,m} g_{lm}(r, r'; k) Y_{lm}(\hat{r}) Y_{lm}(\hat{r}') \tag{39}$$

$$\Longrightarrow (\nabla^2 + k^2)G(\mathbf{x}, \mathbf{x}') = \sum_{l,m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + k^2 \right) g_{lm} Y_{lm}(\hat{r}) Y_{lm}(\hat{r}') \tag{40}$$

$$= \delta^3(\mathbf{x} - \mathbf{x}') = \frac{1}{r^2}\delta(r - r') \sum_{l,m} Y_{lm}(\hat{r}) Y_{lm}(\hat{r}') \tag{41}$$

$$\Longrightarrow \frac{1}{r^2}\delta(r - r') = \left(\frac{\partial^2}{\partial r^2} \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + k^2 \right) g_l(r, r'; k). \tag{42}$$

We find general solutions for the homogeneous equation at $r \neq r'$, and then match the two solutions at $r = r'$.

The general homogeneous solution is

$$g_l = C_1 j_l(kr) = C_2 n_l(kr), \tag{43}$$

where j and n are the spherical Bessel and Neumann functions.

The Helmholtz operator is Hermitian, which means g_l is also, so it must be symmetric under interchange of r and r':

$$g_l = C_l g_<(r_<) g_>(r_>) \tag{44}$$

It must be regular at the origin, so

$$g_<(r_<) = j_l(kr_<). \tag{45}$$

We want only outgoing waves at $r \to \infty$, so

$$g_>(r_>) = j_l(kr_>) + i n_l(kr_>) \equiv h_l^{(1)}(kr_>) \tag{46}$$

Matching derivatives at $(r = r')$:

$$\lim_{\epsilon\to 0} g_l'\big|_{r'-\epsilon}^{r'+\epsilon} = \frac{1}{r^2} \tag{47}$$

$$= C_l j_l(jr') h_l'^{(1)}(kr)\big|_{r'} - C_l j_l'(kr)\big|_{r'} h_l^{(1)}(kr') \tag{48}$$

$$= C_l W(j_l, h_l^{(1)}) \tag{49}$$

where W is the wronskian.

$$\implies C_l \frac{i}{kr^2} = \frac{1}{r^2} \tag{50}$$

Putting everything together:

$$\frac{\hbar^2}{2m}\langle \mathbf{x}|\frac{1}{E - H_0 + i\epsilon}|\mathbf{x}'\rangle = -ik\sum_l \sum_m Y_{lm}(\hat{r})Y_{lm}^*(\hat{r}')j_l(kr_<)h_l^{(1)}(kr_>) \tag{51}$$

(b)

$$\langle \mathbf{x}|Elm(+)\rangle = \langle \mathbf{x}|Elm\rangle + \langle \mathbf{x}|\frac{1}{E - H_0 + i\epsilon}V|Elm(+)\rangle \tag{52}$$

$$= \langle \mathbf{x}|Elm\rangle + \frac{2m}{\hbar^2}\int d^3x' G_+(\mathbf{x},\mathbf{x}')V(r')\langle \mathbf{x}'|Elm(+)\rangle \tag{53}$$

$$= \langle \mathbf{x}|Elm\rangle - \frac{2m}{\hbar^2}\int d^3x' ik \sum_{l,m} Y_l^m(\hat{r})Y_l^{m*}(\hat{r}')j_l(kr_<)h_l^{(1)}(kr_>)V(r')\Psi_{Elm}^+(\mathbf{x}') \tag{54}$$

The free state is

$$\langle \mathbf{x}|Elm\rangle = C_l j_l(kr)Y_l^m(\hat{r}), \tag{55}$$

while a general scattering state can be written as

$$\langle \mathbf{x}|Elm(+)\rangle = C_l A(k,r)Y_l^m(\hat{r}). \tag{56}$$

Pluggin in:

$$\begin{aligned}C_l A(k,r) Y_l^m(\hat{r}) =& C_l j_l(kr) Y_l^m(\hat{r}) \\ &- \frac{2mik}{\hbar^2} \int d^3x' \sum_{l',m'} Y_{l'}^{m'}(\hat{r}) Y_{l'}^{m'*}(\hat{r}') j_{l'}(kr_<) h_{l'}^{(1)}(kr_>) V(r') C_l A(k,r') Y_l^m(\hat{r}')\end{aligned} \tag{57, 58}$$

$$= C_l j_l(kr) Y_l^m(\hat{r}) - \frac{2mik}{\hbar^2} \int dr' r'^2 j_{l'}(kr_<) h_{l'}^{(1)}(kr_>) V(r') C_l A(k,r') Y_l^m(\hat{r}) \tag{59}$$

$$\implies A_l(k;r) = j_l(kr) - \frac{2mik}{\hbar^2} \int_0^\infty j_l(kr_<) h_l^{(1)}(kr_>) V(r') A_l(k;r') r'^2 dr' \tag{60}$$

For $r \to \infty$, then $r_< = r'$, $r_> = r$,

$$A_l(r \to \infty) = j_l(kr) - \frac{2mik}{\hbar^2} h_l^{(1)}(kr) \int_0^\infty j_l(kr') V(r') A_l(k;r') r'^2 dr' \tag{61}$$

$$= e^{i\delta_l} \left[\cos\delta_l j_l(kr) - \sin\delta_l h_l(kr)\right] \tag{62}$$

$$\implies f_l(k) = e^{i\delta_l} \frac{\sin\delta_l}{k} \tag{63}$$

$$= -\left(\frac{2m}{\hbar^2}\right) \int_0^\infty j_l(kr) A_l(k;r) V(r) r^2 dr \tag{64}$$