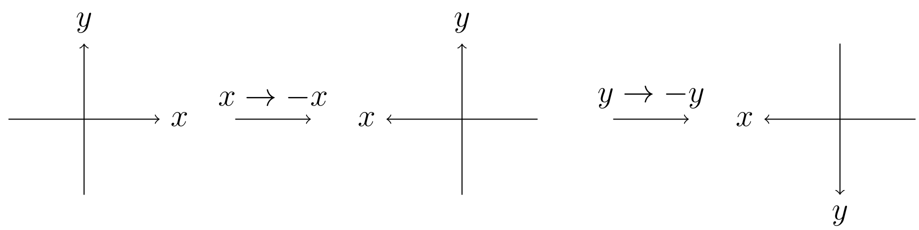
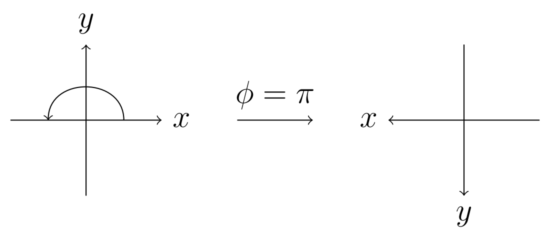
Solutions:

Homework Set 2 — Symmetries 1

Due March 30

1. First consider 2 dimensions (x, y). Taking $x \to -x$ and $y \to -y$ is equivalent to a rotation:





This operation leaves z unchanged, so space inversion ($x \to -x$, $y \to -y$, $z \to -z$) is equivalent to a rotation by π in the (x, y) plane plus a reflection in z (i.e., across the (x, y) plane).

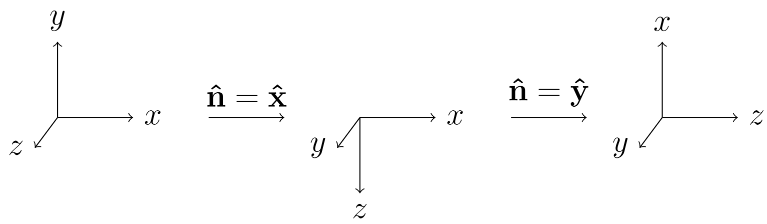
2. (a) Translations commute:

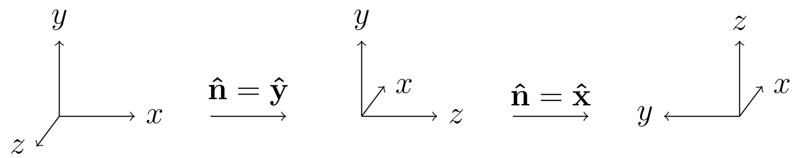
$$\mathcal{T}_{\mathbf{d}'}\mathcal{T}_{\mathbf{d}}\phi(\mathbf{x}) = \mathcal{T}_{\mathbf{d}'}\phi(\mathbf{x}+\mathbf{d}) \tag{1}$$
$$= \phi(\mathbf{x}+\mathbf{d}+\mathbf{d}') \tag{2}$$
$$\mathcal{T}_{\mathbf{d}'}\mathcal{T}_{\mathbf{d}}\phi(\mathbf{x}) = \mathcal{T}_{\mathbf{d}}\phi(\mathbf{x}+\mathbf{d}') \tag{3}$$
$$= \phi(\mathbf{x}+\mathbf{d}'+\mathbf{d}) \tag{4}$$
$$\implies [\mathcal{T}_{\mathbf{d}'}, \mathcal{T}_{\mathbf{d}}] = 0 \tag{5}$$

(b) In general:

$$[\mathcal{D}(\hat{\mathbf{n}}, \phi), \mathcal{D}(\hat{\mathbf{n}}', \phi)] \neq 0 \tag{6}$$

Consider, for example, two rotations by $\pi/2$ with $\hat{\mathbf{n}} = \hat{\mathbf{x}}, \hat{\mathbf{n}}' = \hat{\mathbf{y}}$





(c) Translations and parity do not commute:

$$\begin{aligned} \mathcal{T}_{\mathbf{d}} \pi \psi(\mathbf{x}) &= \mathcal{T}_{\mathbf{d}} \psi(-\mathbf{x}) && (7) \\ &= \psi(-\mathbf{x} + \mathbf{d}) && (8) \\ \pi \mathcal{T}_{\mathbf{d}} \psi(\mathbf{x}) &= \pi \psi(\mathbf{x} + \mathbf{d}) && (9) \\ &= \psi(-\mathbf{x} - \mathbf{d}) && (10) \\ \implies [\mathcal{T}_{\mathbf{d}}, \pi] \neq 0 & && (11) \end{aligned}$$

(d) Parity and rotations commute:

$$\begin{aligned} \pi \mathcal{D}(\hat{\mathbf{n}}, \phi) \psi(\mathbf{x}) &= \pi \psi(\mathbf{x}') && (12) \\ &= \psi(-\mathbf{x}') && (13) \\ \mathcal{D}(\hat{\mathbf{n}}, \phi) \pi \psi(\mathbf{x}) &= \mathcal{D}(\hat{\mathbf{n}}, \phi) \psi(-\mathbf{x})) && (14) \\ &= \psi([-\mathbf{x}]')) && (15) \\ &= \psi(-\mathbf{x}') && (16) \end{aligned}$$

with $\mathbf{x}' = \mathcal{D}(\hat{\mathbf{n}}, \phi)\mathbf{x}$ and $(-\mathbf{x})' = \mathcal{D}(\hat{\mathbf{n}}, \phi)[-\mathbf{x}] = -\mathcal{D}(\hat{\mathbf{n}}, \phi)[\mathbf{x}] = -\mathbf{x}'$
$\implies [\pi, \mathcal{D}(\hat{\mathbf{n}}, \phi)] = 0$

3. Sakurai (1st edition) Eq. 3.7.64:

$$\mathcal{Y}_l^{j=l\pm 1/2, m} = \frac{1}{\sqrt{2l+1}} \begin{bmatrix} \pm\sqrt{l \pm m + \frac{1}{2}} Y_l^{m-1/2}(\theta, \phi) \\ \sqrt{l \mp m + \frac{1}{2}} Y_l^{m+1/2}(\theta, \phi) \end{bmatrix} \qquad (17)$$

(a) For $l = 0$, only $j = 1/2$ (upper sign) is possible, so

$$\mathcal{Y}_{l=0}^{j=1/2,m=1/2} = \frac{1}{\sqrt{4\pi}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{18}$$

(b)

$$\sigma \cdot \mathbf{x} \frac{1}{\sqrt{4\pi}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{4\pi}} \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{19}$$

$$= \frac{1}{\sqrt{4\pi}} \begin{bmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{bmatrix} \tag{20}$$

$$= -r \begin{bmatrix} -Y_1^0(\theta, \phi)/\sqrt{3} \\ \left(\frac{2}{3}\right)^1 /2 Y_1^1(\theta, \phi) \end{bmatrix}, \tag{21}$$

where we recall $Y_1^0 = (3/4\pi)^1/2 \cos\theta$ and $Y_1^1 = -(3/8\pi)^1/2 \sin\theta e^{i\phi}$. Compare with $\mathcal{Y}_l^{j,m}$ in Eq. (17), we see that m must be 1/2, l must be 1. Take the lower sign, hence $j = l - 1/2 = 1/2$. So Eq. (19) becomes

$$\sigma \cdot \mathbf{x} \frac{1}{\sqrt{4\pi}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(-\frac{r}{\sqrt{3}}\right) \begin{bmatrix} -\sqrt{1 - \frac{1}{2} + \frac{1}{2}} Y_1^0 \\ \sqrt{1 + \frac{1}{2} + \frac{1}{2}} Y_1^1 \end{bmatrix} \tag{22}$$

$$= -r \mathcal{Y}_{l=1}^{j=1/2,m=1/2} \tag{23}$$

Conclusion: Apart from $-r$, we get $\mathcal{Y}_l^{j,m}$ with l changed ($l = 0 \to l = 1$) and j, m both unchanged from Eq. (18)

(c) The result obtained in 3b is not surprising: $\mathbf{S} \cdot \mathbf{x}$ is scalar (spherical tensor of rank 0) under rotations. By the Wigner-Eckart theorem it cannot change j and m. However, under parity $\mathbf{S} \cdot \mathbf{x}$ is odd (pseudoscalar). So it connects even parity with odd parity states, and we note that $l = 0$ and $l = 1$ states have opposite parity.

4. $\mathbf{S} \cdot \mathbf{p}$ is invariant under rotations but changes sign under parity. So it is a pseudoscalar. Since $\delta^3(\mathbf{x})$ is a scalar, V is a pseudoscalar. So V can connect l odd and l even, but cannot change j, m.

   From first order perturbation theory we have

$$C_{n'l'j'm'} = \frac{\langle n'l'j'm'|V|nljm\rangle}{E_{nlj} - E_{n'l'j'}}, \tag{24}$$

where $l' = l \pm 1$ (note that $|\Delta l| \geq 2$ is impossible because j must remain the same) and $m' = m$, $j' = j$.

The wave function for $|nljm\rangle$ can be written as $R_{nlj}\mathcal{Y}_l^{j=l\pm1/2,m}$ where $\mathcal{Y}_l^{j,m}$ is the spin angular function and for low Z, R_{nlj} has no dependence on j. So $\langle n'l'j'm'|V|nljm\rangle$ becomes

$$\langle n'l'j'm'|V|nljm\rangle$$
$$= \lambda \int d^3x R_{n'l'j'}(r)\mathcal{Y}_{l'}^{j'=l'\pm1/2,m} \left[\delta^{(3)}(\mathbf{x})S\cdot(-i\hbar\overrightarrow{\nabla}) + (-i\hbar\overleftarrow{\nabla})\cdot S\delta^{(3)}(\mathbf{x})\right]\cdot R_{nlj}(r)\mathcal{Y}_l^{j=l\pm1/2,m} \tag{25}$$

where $(-i\hbar\overleftarrow{\nabla})$ in the second term operates on the wavefunction to the left.

Because of the δ function, the matrix element vanishes unless $R_{nlj}(r)$ or $R_{n'l'j'}(r)$ is finite at $r = 0$. Thus we must have $S_{1/2}$ or $P_{1/2}$ for $|nljm\rangle$ to obtain non-vanishing contributions to $C_{n'l'j'm'}$