Homework Set 2 — Symmetries 1

Due Apriil 11, 2021

1. Parity is often treated as a mirror reflection. This is certainly true in 1 dimension, where $x \to -x$ may be viewed as the effect of reflecting through a (point) mirror at the origin. In three dimensions when we use a plane mirror (say lying on the $x - y$ plane), only one (z) coordinate gets reversed, whereas the parity transformation reverses all three coordinates.

   Verify that reflection in a mirror in the $x - y$ plane is the same as parity followed by a π (180°) rotation about the z axis. Since rotational invariance holds for weak interactions, noninvariance under mirror reflection implies noninvariance under parity.

2. Let $\mathcal{T}_{\mathbf{d}}$ denote the translation operator (with displacement vector $\mathbf{d}$); $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ the rotation operator ($\hat{\mathbf{n}}$ and ϕ are the axis and angle of rotation, respectively); and π the parity operator. Which, if any, of the following pairs commute? Why?

   (a) $\mathcal{T}_{\mathbf{d}}$ and $\mathcal{T}_{\mathbf{d}'}$ (with **d** and **d'** in different directions)

   (b) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and $\mathcal{D}(\hat{\mathbf{n}}', \phi)$ ($\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ in different directions)

   (c) $\mathcal{T}_{\mathbf{d}}$ and π

   (d) $\mathcal{D}(\hat{\mathbf{n}}, \phi)$ and π

3. A spin $\frac{1}{2}$ particle is bound to a fixed center by a spherically symmetric potential.

   (a) Write down the spin angular function $\mathcal{Y}_{l=0}^{j=1/2,m=1/2}$ (see, e.g., Sakurai Chapter 3)

   (b) Express $(\sigma \cdot \mathbf{x})\mathcal{Y}_{l=0}^{j=1/2,m=1/2}$ in terms of some other $\mathcal{Y}_{l}^{j,m}$.

   (c) Show that your result in 3b is understandable in view of the transformation properties of the operator $\mathbf{S} \cdot \mathbf{x}$ under rotations and under space inversion (parity)

4. Because of weak (neutral-current) interactions there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda \left[\delta^{(3)}(\mathbf{x}) \mathbf{S} \cdot \mathbf{p} + \mathbf{S} \cdot \mathbf{p} \delta^{(3)}(\mathbf{x}) \right], \tag{1}$$

where $\mathbf{S}$ and $\mathbf{p}$ are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by $|n, l, j, m\rangle$ actually contains very tiny contributions from other eigenstates as follows:

$$|n, l, j, m\rangle \to |n, l, j, m\rangle + \sum_{n'l'j'm'} C_{n'l'j'm'} |n', l', j', m'\rangle. \tag{2}$$

On the basis of symmetry considerations *alone*, what can you say about (n', l', j', m'), which give rise to nonvanishing contributions? Suppose the radial wave functions and the energy levels are all known. Indicate how you may calculate $C_{n'l'j'm'}$. Do we get further restrictions on (n', l', j', m')?