Solutions:
Homework Set 1 — Identical Particles

Due April 4, 2021

1. Show the following:

   (a) $\left[\Omega_1^{(1)} \otimes \mathbb{1}^{(2)}, \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)}\right] = 0$ for any $\Omega_1^{(1)}$ and $\Lambda_2^{(2)}$ (operaters of particle 1 commute with those of particle 2).

$$\begin{aligned}
\left[\Omega_1^{(1)} \otimes \mathbb{1}^{(2)}, \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)}\right] |\omega_1\rangle \otimes |\omega_2\rangle &= \Omega_1^{(1)} \otimes \mathbb{1}^{(2)}\mathbb{1}^{(1)} \otimes \Lambda_2(2)|\omega_1\rangle \otimes |\omega_2\rangle \\
&\quad - \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)}\Omega_1^{(1)} \otimes \mathbb{1}^{(2)}|\omega_1\rangle \otimes |\omega_2\rangle \qquad (1)\\
&= \Omega_1^{(1)} \otimes \mathbb{1}^{(2)}|\mathbb{1}^{(1)}\omega_1\rangle \otimes |\Lambda_2^{(2)}\omega_2\rangle \\
&\quad - \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)}|\Omega_1^{(1)}\omega_2\rangle \otimes |\mathbb{1}^{(2)}\omega_2\rangle \qquad (2)\\
&= |\Omega_1^{(1)}\omega_1\rangle \otimes |\mathbb{1}^{(1)}\Lambda_2^{(2)}\omega_2\rangle - |\mathbb{1}^{(1)}\Omega_1^{(1)}\omega_1\rangle \otimes |\Lambda_2^{(2)}\omega_2\rangle \qquad (3)\\
&= |\Omega_1^{(1)}\omega_1\rangle \otimes |\Lambda_2^{(2)}\omega_2\rangle - |\Omega_1^{(1)}\omega_1\rangle \otimes |\Lambda_2^{(2)}\omega_2\rangle \qquad (4)\\
&= 0 \qquad (5)
\end{aligned}$$

   (b) $\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)\left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right) = (\Omega\theta)_1^{(1)} \otimes (\Gamma\Lambda)_2^{(2)}$

$$\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)\left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right)|\omega_1\rangle \otimes |\omega_2\rangle = \left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)|\theta_1^{(1)}\omega\rangle \otimes |\Lambda_2^{(2)}\omega_2\rangle \tag{6}$$

$$= |\Omega_1^{(1)}\theta_1^{(1)}\omega_1\rangle \otimes |\Gamma_2^{(2)}\Lambda_2^{(2)}\omega_2\rangle \tag{7}$$

$$= \left(\Omega_1^{(1)}\theta_1^{(1)}\right) \otimes \left(\Gamma_2^{(2)}\Lambda_2^{(2)}\right)|\omega_1\rangle \otimes |\omega_2\rangle \tag{8}$$

$$= \left[(\Omega\theta)^{(1)} \otimes (\Gamma\Lambda)^{(2)}\right]|\omega_1\rangle \otimes |\omega_2\rangle \tag{9}$$

$$\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)\left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right) = (\Omega\theta)^{(1)} \otimes (\Gamma\Lambda)^{(2)} \tag{10}$$

(c) If $\left[\Omega_1^{(1)}, \Lambda_1^{(1)}\right] = \Gamma_1^{(1)}$,

then $\left[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right] = \Gamma_1^{(1)} \otimes \mathbb{1}^{(2)}$,

and similarly with $1 \to 2$.

$$\left[\Gamma_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right]|\omega_1\rangle \otimes |\omega_2\rangle = \left[\Gamma_1^{(1)} \otimes \mathbb{1}^{(2)}, \Lambda_1^{(1)} \otimes \mathbb{1}^{(2)}\right]|\omega_1\rangle \otimes |\omega_2\rangle \tag{11}$$

$$= |\left[\Gamma_1^{(1)}, \Lambda_1^{(1)}\right]\omega_1\rangle \otimes |\mathbb{1}^{(2)}\omega_2\rangle \tag{12}$$

$$= |\Gamma_1^{(1)}\omega_1\rangle \otimes |\mathbb{1}^{(2)}\omega_2\rangle \tag{13}$$

$$= \Gamma_1^{(1)} \otimes \mathbb{1}^{(2)}|\omega_1\rangle \otimes |\omega_2\rangle \tag{14}$$

$$\left[\Gamma_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right] = \Gamma_1^{(1)} \otimes \mathbb{1}^{(2)} \tag{15}$$

(d) $\left(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)}\right)^2 = (\Omega_1^2)^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes (\Omega_1^2)^{(1)} + 2\Omega_1^{(1)} \otimes \Omega_2^{(2)}$

$$
\begin{aligned}
\left(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)}\right)^2 &= \left(\Omega_1^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(2)} \otimes \Omega_2^{(1)}\right)^2 |\omega_1\rangle \otimes \omega_2\rangle \qquad (16) \\
&= \left(\Omega_1^{(1)} \otimes \mathbb{1}^{(2)}\right)^2 |\omega_1\rangle \otimes \omega_2\rangle \\
&\quad + \Omega_1^{(1)} \otimes \mathbb{1}^{(2)}\mathbb{1}^{(1)} \otimes \Omega_2^{(2)} |\omega_1\rangle \otimes \omega_2\rangle \\
&\quad + \mathbb{1}^{(1)} \otimes \Omega_2^{(2)}\Omega_1^{(1)} \otimes \mathbb{1}^{(2)} |\omega_1\rangle \otimes \omega_2\rangle \\
&\quad + \left(\mathbb{1}^{(1)} \otimes \Omega_2^{(2)}\right)^2 |\omega_1\rangle \otimes \omega_2\rangle \qquad (17) \\
&= |(\Omega_1^2)^{(1)}\omega_1\rangle \otimes |\mathbb{1}^{(2)}\omega_2\rangle + |\Omega_1^{(1)}\omega_1\rangle \otimes |\Omega_2^{(2)}\omega_2\rangle \\
&\quad + |\Omega_1^{(1)}\omega_1\rangle \otimes |\Omega_2^{(2)}\omega_2\rangle + |\mathbb{1}^{(1)}\omega_1\rangle \otimes |(\Omega_2^2)^{(2)}\omega_2\rangle \\
&= \left(\left(\Omega_1^2\right)^{(1)} \otimes \mathbb{1}^{(2)} + 2\Omega_1^{(1)} \otimes \Omega_2^{(2)} + \mathbb{1}^{(1)} \otimes \left(\Omega_1^2\right)^{(1)}\right) |\omega_1\rangle \otimes \omega_2\rangle \qquad (18)
\end{aligned}
$$

$$
\left(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)}\right)^2 = \left(\Omega_1^2\right)^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes \left(\Omega_1^2\right)^{(1)} + 2\Omega_1^{(1)} \otimes \Omega_2^{(2)} \qquad (19)
$$

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2. Consider a two-particle Hilbert space, constructed from single-particle spaces spanned by two basis vectors $|+\rangle$ and $|-\rangle$. Let

$$
\sigma_1^{(1)} = \begin{matrix} & + & - \\ + & a & b \\ - & c & d \end{matrix} \quad \text{and} \quad \sigma_2^{(2)} = \begin{matrix} & + & - \\ + & e & f \\ - & g & h \end{matrix} \qquad (20)
$$

be operators in spaces (1) and (2), respectively (the $\pm$ signs label the basis vectors, so $b = \langle +|\sigma_1^{(1)}|-\rangle$, etc.). The space is spanned by four vectors $|+\rangle \otimes |+\rangle$, $|+\rangle \otimes |-\rangle$, $|-\rangle \otimes |+\rangle$, $|-\rangle \otimes |-\rangle$. Show that

$$
\sigma_1^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes \mathbb{1}^{(2)} = \begin{matrix} & ++ & +- & -+ & -- \\ ++ & a & 0 & b & 0 \\ +- & 0 & a & 0 & b \\ -+ & c & 0 & d & 0 \\ -- & 0 & c & 0 & d \end{matrix} \qquad (21)
$$

$$\sigma_2^{(1)\otimes(2)} = \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} \tag{22}$$

$$(\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes \sigma_2^{(2)} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix} \tag{23}$$

Show Eq. (23) in two ways, by taking the matrix product of $\sigma_1^{(1)\otimes(2)}$ and $\sigma_2^{(1)\otimes(2)}$ and by directly computing the matrix elements of $(\sigma_1\sigma_2)^{(1)\otimes(2)}$.

3. N identical spin $\frac{1}{2}$ particles are subjected to a one-dimensional simple harmonic oscillator potential. What is the ground state energy? What is the Fermi energy? What happens when $N \to \infty$

For a single particle in a 1D SHO potential the energy spectrum is

$$E_n^{(1)} = \hbar\omega\left(n + \frac{1}{2}\right); \qquad n = 0, 1, 2, \ldots \tag{24}$$

Each energy state can accommodate a maximum 2 spin $\frac{1}{2}$ particles (one for each possible spin state, $2s + 1$). That is, as fermions, if they are in the same spatial state (and therefore in a spatial state that is symmetric under particle interchange), they must be in an antisymmetric spin state. The maximum number of particles in a completely antisymmetric spin state is equal to the number of spin states. For spin-1/2 electrons, there can be only 2 particles in an antisymmetric spin state.
E.g., for 3 particles, two can be in the lowest energy $E_0^{(1)}$ and the third must be in the next higher state $E_1^{(1)}$, and the ground state is

$$E_0^{(3)} = 2\left(\frac{1}{2}\hbar\omega\right) + 1\left(\frac{3}{2}\hbar\omega\right) = \frac{5}{2}\hbar\omega \tag{25}$$

For an arbitrary number N particles, when N is even, each occupied state contains two particles and the total ground state energy is

$$E_0^{(N)} = 2\sum_{n=0}^{\frac{N}{2}-1} \hbar\omega(n+\frac{1}{2}) \tag{26}$$

$$= \frac{\hbar\omega}{4}N^2. \tag{27}$$

When N is odd, the highest energy single-particle state is singly occupied:

$$E_0^{(N)} = 2\sum_{n=0}^{\frac{N-3}{2}} \hbar\omega(n+\frac{1}{2}) + \hbar\omega(\frac{N}{2}-\frac{1}{2}+\frac{1}{2}) \tag{28}$$

$$= \frac{\hbar\omega}{4}(N^2+1). \tag{29}$$

or, summarizing

$$E_0^{(N)} = \begin{cases} \frac{N^2}{4}\hbar\omega, & N = \text{even} \\ \frac{N^2+1}{4}\hbar\omega, & N = \text{odd} \end{cases} \tag{30}$$

The Fermi energy is the highest occupied single-particle state in the N-particle ground state. The highest occupied state is $E^{(1)}_{(N-1)/2}$ for odd N and $E^{(1)}_{(N-2)/2}$ for even N. So the Fermi energy is

$$E_F^{(N)} = \begin{cases} \frac{N-1}{2}\hbar\omega & N = \text{even} \\ \frac{N}{2}\hbar\omega & N = \text{odd} \end{cases} \tag{31}$$

In the limit $N \to \infty$, we have

$$\lim_{N\to\infty} E_0^{(N)} = \frac{N^2}{4}\hbar\omega, \tag{32}$$

$$\lim_{N\to\infty} E_F^{(N)} = \frac{N}{2}\hbar\omega. \tag{33}$$

There was a question on Wednesday about how it is possible to construct fermionic states with more than 2 electrons, if the spatial state is not fully antisymmetric. Here is an explicit example of the lowest-energy 3-electron state in this problem. An example of the notation is the first term $|001 \uparrow\downarrow\uparrow\rangle$, which means particle 1 has energy E_0 and spin up ($\uparrow$), particle 2 has energy E_0 and spin down ($\downarrow$), and particle 3 has energy E_1 and spin up ($\uparrow$).

$$|\psi_{\rm GS}\rangle = \Big[|001 \uparrow\downarrow\uparrow\rangle - |001 \downarrow\uparrow\uparrow\rangle\Big] - \Big[|100 \uparrow\downarrow\uparrow\rangle - |100 \uparrow\uparrow\downarrow\rangle\Big] - \Big[|010 \uparrow\uparrow\downarrow\rangle - |010 \downarrow\uparrow\uparrow\rangle\Big]$$

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4. Two nonidentical spin 1 particles with no orbital angular momenta (that is, *s*-states for both) can form j=0, j=1, and j=2 states. Suppose, however, that the two particles are *identical*. What restrictions do we get?

Spin 1 particles are bosons and must have symmetric wavefunction. They are in the same orbital state, and so they must also be in a symmetric spin state.

The $j = 0$ state and the (5) $j = 2$ states are symmetric with respect to interchange of particles, while the (3) $j = 1$ states are antisymmetric. Thus, only states with $j = 0$ or $j = 2$ are allowed.

One can see this explicitly by constructing the full states. For example, the stretch state $|j = 2, m = 0\rangle$ must be symmetric, $|+\rangle|+\rangle$, and the other $j = 2$ states can be obtained from the lowering operator $S_- = S_{1-} + S_{2-}$:

$$|j = 2, m = 2\rangle = |+\rangle|+\rangle \tag{34}$$

$$|j = 2, m = 1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|0\rangle + |0\rangle|+\rangle\right) \tag{35}$$

$$|j = 2, m = 0\rangle = \frac{1}{\sqrt{6}}\left(2|0\rangle|0\rangle + |-\rangle|+\rangle - |+\rangle|-\rangle\right) \tag{36}$$

$$|j = 2, m = -1\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle|0\rangle + |0\rangle|-\rangle\right) \tag{37}$$

$$|j = 2, m = -2\rangle = |-\rangle|-\rangle. \tag{38}$$

$$\tag{39}$$

These are all symmetric.

The $|j = 1, m = 1\rangle$ state must be orthogonal to the $|j = 2, m = 1\rangle$ state, and can be similarly lowered to obtain the other states:

$$|j = 1, m = 1\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle|0\rangle - |0\rangle|+\rangle\right) \tag{40}$$

$$|j = 1, m = 0\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle|-\rangle - |-\rangle|+\rangle\right) \tag{41}$$

$$|j = 1, m = -1\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle|-\rangle - |-\rangle|0\rangle\right) \tag{42}$$

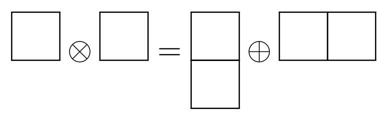
These states are all antisymmetric.

The $|j = 0, m = 0\rangle$ state must be orthogonal to the other $m = 0$ states:

$$|j = 0, m = 0\rangle = \frac{1}{\sqrt{3}} \left(|+\rangle|-\rangle - |0\rangle|0\rangle + |-\rangle|+\rangle\right) \tag{43}$$

This state is symmetric with respect to particle interchange.

Or, we can see



(44)

$$3 \times 3 = 3 + 6 \tag{45}$$

There are 3 antisymmetric states and 6 symmetric states. We know there must be 5 $j = 2$ states, 3 $j = 1$ states, and 1 singlet state. So the antisymmetric states must represent the $j = 1$ triplet, and are not allowed.

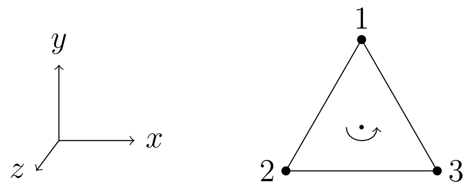
5. Discuss what would happen to the energy levels of a helium atom if the electron were a spinless boson.

For two spinless bosons, the orbital wavefunction is necessarily symmetric under interchange of the particles

If we neglect spin interactions, then the energy levels would be the same for $\frac{1}{2}$ or spin 0 “electrons”. However, excited states with antisymmetric wavefunctions are not allowed, and would disappear from the energy spectrum. Specifically, all the spin-triplet states (orthohelium) would disappear.

6. Three identical spin 0 particles are situated at the corners of an equilateral triangle. Let

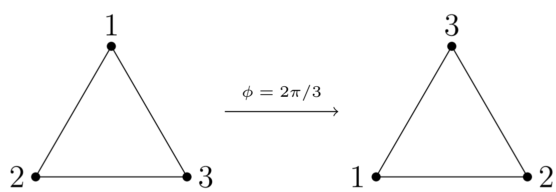
us define the z-axis to go through the center and in the direction normal to the plane of the triangle. The whole system is free to rotate about the z-axis. Using statistics considerations, obtain restrictions on the magnetic quantum numbers corresponding to J_z.



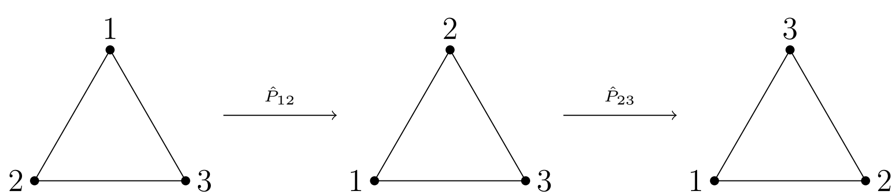
The rotation operator around the z-axis by angle ϕ can be written

$$D(\phi) = e^{-i\hat{J}_z\phi/\hbar}. \tag{46}$$

Rotations by multiplies of $\phi = 2\pi/3$ are equivalent to a series of particle exchanges, and must therefore leave the wavefunction invariant (since the particles are bosons). E.g., a rotation by $2\pi/3$



is equivalent to the consecutive interchange $\hat{P}_{23}\hat{P}_{12}$



So it must be an eigenstate of $D(\phi = 2\pi/3)$ with eigenvalue 1. It must then be an eigenstate of $\hat{J}_z$:

$$\begin{aligned} D(\phi = 2\pi/3)|\alpha\rangle &= e^{-i\hat{J}_z\phi/\hbar}|\alpha\rangle && (47)\\ &= e^{-im*2\pi/3}|\alpha\rangle && (48)\\ &= |\alpha\rangle, && (49) \end{aligned}$$

and the state must have $m = 0, \pm 3, \pm 6, \ldots$, or $\frac{m}{3} \in \mathbb{Z}$

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7. Consider three weakly interacting, identical spin 1 particles

   (a) Suppose the space part of the state vector is known to be symmetric under interchange of *any* pair. Using notation $|+\rangle|0\rangle|+\rangle$ for particle 1 in m_s=+1, particle 2 in m_s=0, particle 3 in m_s=+1, and so on, construct the normalized spin states in the following three cases:

      i. All three of them in $|+\rangle$.

      ii. *Two* of them in $|+\rangle$, one in $|0\rangle$.

      iii. All three in different spin states

      What is the total spin in each case?

   (b) Attempt to do the same problem when the space part is antisymmetric under interchange of any pair.

(a) The space part of the state vector is symmetric $\Longrightarrow$ the spin part must be symmetric under interchange of any pair

   i. The symmetric state is trivially

$$|+\rangle|+\rangle|+\rangle \tag{50}$$

   This has total spin $s = 3$ (and $m = 3$)

   ii. The only completely symmetric combination is

$$\frac{1}{\sqrt{3}}\left(|+\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|+\rangle + |0\rangle|+\rangle|+\rangle\right) \tag{51}$$

   This is the $s = 3$, $m = 2$ state.

   iii. The completely symmetric combination here is

$$\frac{1}{\sqrt{6}}\left(|+\rangle|0\rangle|-\rangle + |+\rangle|-\rangle|0\rangle + |0\rangle|+\rangle|-\rangle + |-\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|+\rangle + |-\rangle|0\rangle|+\rangle\right) \tag{52}$$

   This is not an eigenstate of S^2, and so does not have a definite total spin. It is a linear combination of $m = 0$ states with different s. One can check this by acting on the state with the S^2 operator, and note that an extra term appears:

$$S^2|\Psi\rangle = 12|\Psi\rangle + \frac{24}{\sqrt{6}}|0\rangle|0\rangle|0\rangle \tag{53}$$

(b) For 7(a)iii, there is one completely antisymmetric state:

$$\frac{1}{\sqrt{6}}\left(|+-0\rangle - |+-0\rangle + |0-+\rangle - |0+-\rangle + |-+0\rangle - |-0+\rangle\right) \tag{54}$$

This is the spin singlet state $s = 0$.

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8. Suppose the electron were a spin $\frac{3}{2}$ particle obeying Fermi-Dirac statistics. Write the configuration of a hypothetical Ne (Z=10) atom made up of such "electrons" [that is, the analog of $(1s)^2(2s)^2(2p)^6$]. Show that the configuration is highly degenerate. What is the ground state (the lowest term) of the hypothetical Ne atom in spectroscopic notation ($^{2S+1}L_J$, where S, L, and J stand for the total spin, the total orbital angular momentum, and the total angular momentum, respectively) when exchange splitting and spin-orbit splitting are taken into account?

For each spin $\frac{3}{2}$ electron there are four possible spin orientations. Therefore it is possible to put four electrons in each spatial state.

The lowest energy configuration is then:

$$(1s)^4(2s)^4(2p)^2 \tag{55}$$

This state is degenerate because each of the two electrons in the p state can have $m_l = 1, 0, -1$ and $m_s = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

A single particle has 12 possible states, and so two indistinguishable can occupy these states a total of

$$\binom{12}{2} = \frac{12!}{2!(12-2)!} = 66 \tag{56}$$

different ways, compared to the single ground state configuration of the standard case.

All of these are allowed, since we can make an antisymmetric state for any two distinct choices for $(m_l,\, m_s)$:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_{m_l,m_s}\rangle|\Psi_{m_l',m_s'}\rangle - |\Psi_{m_l',m_s'}\rangle|\Psi_{m_l,m_s}\rangle\right) \tag{57}$$

This degeneracy is broken when you take into account spin-orbit and exchange splitting. The piece of the Hamiltonian responsible for the spin-orbit interaction is proportional to

$$\hat{L} \cdot \hat{S} = \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2) \tag{58}$$

$\therefore$ the lowest energy state is when S and L are maximized while J is minimized.

The total spin of two spin $\frac{3}{2}$ particles can be S = 3,2,1, or 0, while the total orbital angular momentum can be L = 2, 1, or 0. $(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)\Psi$ is minimized for S=3, L=2, J=1. However, this is not allowed, because the wavefunction must be antisymmetric.

With this restriction the minimum is for S=3, L=1, J=2. This also agrees with the effects of exchange splitting – the lowest energy states have a symmetric spin state and antisymmetric spatial state. The new ground state is written $^7P_2$