Homework Set 6 — Quantum Information

Due June 15, 2020

1. The trace of an operator is defined as $\mathrm{Tr}\{A\} = \sum_m \langle m|A|m\rangle$, where $\{|m\rangle\}$ is an arbitrary orthonormal basis set. Introduce a second basis set, and use it to prove that the trace is independent of the choice of basis.

2. Prove the linearity of the trace operation by proving

$$\mathrm{Tr}\{aA + bB\} = a\mathrm{Tr}\{A\} + b\mathrm{Tr}\{B\} \tag{1}$$

3. Prove the cyclic property of the trace by proving

$$\mathrm{Tr}\{ABC\} = \mathrm{Tr}\{BCA\} = \mathrm{Tr}\{CAB\} \tag{2}$$

4. Which of the following density matrices correspond to a pure state?

$$\rho_1 = \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{5}{7} \end{pmatrix} \tag{3}$$

$$\rho_2 = \begin{pmatrix} \frac{1}{4} & i\frac{\sqrt{3}}{4} \\ -i\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix} \tag{4}$$

$$\rho_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$\rho_4 = \begin{pmatrix} \frac{1}{5} & \frac{\sqrt{2}}{5} \\ \frac{\sqrt{2}}{5} & \frac{4}{5} \end{pmatrix} \tag{6}$$

5. The density matrix evolves with time. Derive the equation of motion

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] \tag{7}$$

using the Schrodinger equation and the most general form of the density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{8}$$

6. A spin-$\frac{1}{2}$ particle is in a statistical ensemble with a 50% probability to be in the $|+_z\rangle$ state (the eigenstate of S_z with eigenvalue $\hbar/2$) a 50% chance to be in $|+_x\rangle$ (the eigenstate of S_x with eigenvalue $\hbar/2$). [Note that these states are not orthogonal. Don't worry about that yet.] Use the standard procedure,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{9}$$

to write this density operator in terms of states in the $|\pm_z\rangle$ basis, and then as a matrix using this basis. Use the density matrix to compute the probability that a measurement of the z-component of spin will return value $+\hbar/2$.

Now solve the eigenvalue/eigenvector problem for the density matrix. The eigenstates you find are eigenstates of spin along a definite axis — that is, eigenstatss of $S \cdot \hat{n}$ for some unit vector $\hat{n}$. Find $\hat{n}$. What is the entropy of the density matrix, and is it the same as you would guess at the beginning, knowing that there is a 50% chance to be in each of two states? What would be a more proper description of the density matrix, in terms of probabilities to be in orthogonal states?