Homework Set 4 — Scattering 1

Due May 11, 2020

1. Given a cross section of
$$\frac{d\sigma}{d\Omega} = 10^{-34} \sin^2\theta \text{ cm}^2, \tag{1}$$
compute the number of particles per unit time scattering into a detector with circular cross-section area 10 m^2 located a distance 100 m from the scattering center off-axis at an angle of 30° from the beam axis that is exposed to a beam of 2×10^{32} particles per unit time per cm^2

2. The Lippmann-Schwinger formalism can also be applied to a *one*-dimensional transmission-reflection problem with a finite-range potential, $V(x) \neq 0$ for $0 < |x| < a$ only.

   (a) Suppose we have an incident wave coming from the left: $\langle x|\phi\rangle = e^{ikx}/\sqrt{2\pi}$. How must we handle the singular $1/(E - H_0)$ operator if we are to have a transmitted wave only for $x > a$ and a reflected wave and the original wave for $x < -a$? Is the $E \to E + i\epsilon$ prescription still correct? Obtain an expression for the appropriate Green's function and write an integral equation for $\langle x|\psi^{(+)}\rangle$.

   (b) Consider the special case of an attractive δ-function potential
   $$V = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x), \quad (\gamma > 0). \tag{2}$$
   Solve the integral equation to obtain the transmission and reflection amplitudes.

   (c) The one-dimensional δ-function potential with $\gamma > 0$ admits one (and only one) bound state for any value of γ. Show that the transmission and reflection amplitudes you computed have bound-state poles at the expected positions when k is regarded as a complex variable.

3. Consider the 1D step-up potential

$$V(r) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases} \tag{3}$$

with $V_0 > 0$. A particle with kinetic energy $E > V_0$ is incident from the left.

(a) Find the intensity of the reflected (R) and transmitted (T) waves.

(b) Find the currents (i.e., the intensities multiplied by the velocities) J_R and J_T of the reflected and transmitted waves and their sum $J_R + J_T$.

4. (a) Prove

$$\frac{\hbar^2}{2m}\langle \mathbf{x}|\frac{1}{E - H_0 + i\epsilon}|\mathbf{x}'\rangle = -ik\sum_l\sum_m Y_l^m(\hat{r})Y_l^{m*}(\hat{r}')j_l(kr_<)h_l^{(1)}(kr_>), \tag{4}$$

where $r_<(r_>)$ stands for the smaller (larger) of r and r'.

(b) For spherically symmetric potentials, the Lippmann-Schwinger equation can be written for *spherical* waves:

$$|Elm(+)\rangle = |Elm\rangle + \frac{1}{E - H_0 + i\epsilon}V|Elm(+)\rangle. \tag{5}$$

Using (4), show that this equation, written in the **x**-representation, leads to an equation for the radial function, $A_l(k;r)$)k, as follows:

$$A_l(k;r) = j_l(kr) - \frac{2mik}{\hbar^2}\int_0^\infty j_j(kr_<)h_l^{(1)}(kr_>)V(r')A_l(k;r')r'^2dr'. \tag{6}$$

By taking r very large, also obtain

$$f_l(k) = e^{i\delta_l}\frac{\sin\delta_l}{k} \tag{7}$$

$$= -\left(\frac{2m}{\hbar^2}\right)\int_0^\infty j_l(kr)A_l(k;r)V(r)r^2dr \tag{8}$$