Homework Set 1 — Identical Particles

Due March 23, 2020

1. Show the following:

   (a) $\left[\Omega_1^{(1)} \otimes \mathbb{1}^{(2)}, \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)}\right] = 0$ for any $\Omega_1^{(1)}$ and $\Lambda_2^{(2)}$
   (operaters of particle 1 commute with those of particle 2).

   (b) $\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right)\left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right) = (\Omega\theta)_1^{(1)} \otimes (\Gamma\Lambda)_2^{(2)}$

   (c) If $\left[\Omega_1^{(1)}, \Lambda_1^{(1)}\right] = \Gamma_1^{(1)}$,
   then $\left[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right] = \Gamma_1^{(1)} \otimes \mathbb{1}^{(2)}$,
   and similarly with $1 \to 2$.

   (d) $\left(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)}\right)^2 = (\Omega_1^2)^{(1)} \otimes \mathbb{1}^{(2)} + \mathbb{1}^{(1)} \otimes (\Omega_1^2)^{(1)} + 2\Omega_1^{(1)} \otimes \Omega_2^{(2)}$

2. Consider a two-particle Hilbert space, constructed from single-particle spaces spanned by two basis vectors $|+\rangle$ and $|-\rangle$. Let

$$\sigma_1^{(1)} = \begin{array}{c} \\ + \\ - \end{array}\!\!\begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \quad \text{and} \quad \sigma_2^{(2)} = \begin{array}{c} \\ + \\ - \end{array}\!\!\begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \end{array} \tag{1}$$

be operators in spaces (1) and (2), respectively (the $\pm$ signs label the basis vectors, so $b = \langle +|\sigma_1^{(1)}|-\rangle$, etc.). The space is spanned by four vectors $|+\rangle \otimes |+\rangle$, $|+\rangle \otimes |-\rangle$, $|-\rangle \otimes |+\rangle$, $|-\rangle \otimes |-\rangle$. Show that

$$\sigma_1^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes \mathbb{1}^{(2)} = \begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array}\!\!\begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \end{array} \tag{2}$$

$$\sigma_2^{(1)\otimes(2)} = \begin{pmatrix} e & f & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} \tag{3}$$

$$(\sigma_1\sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes \sigma_2^{(2)} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix} \tag{4}$$

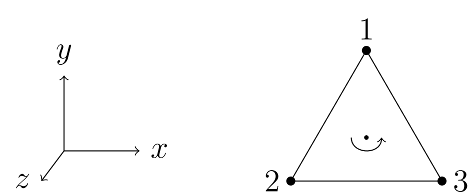
Show Eq. (4) in two ways, by taking the matrix product of $\sigma_1^{(1)\otimes(2)}$ and $\sigma_2^{(1)\otimes(2)}$ and by directly computing the matrix elements of $(\sigma_1\sigma_2)^{(1)\otimes(2)}$.

3. N identical spin $\frac{1}{2}$ particles are subjected to a one-dimensional simple harmonic oscillator potential. What is the ground state energy? What is the Fermi energy? What happens when $N \to \infty$

4. Two nonidentical spin 1 particles with no orbital angular momenta (that is, s-states for both) can form j=0, j=1, and j=2 states. Suppose, however, that the two particles are *identical*. What restrictions do we get?

5. Discuss what would happen to the energy levels of a helium atom if the electron were a spinless boson.

6. Three identical spin 0 particles are situated at the corners of an equilateral triangle. Let us define the z-axis to go through the center and in the direction normal to the plane of the triangle. The whole system is free to rotate about the z-axis. Using statistics considerations, obtain restrictions on the magnetic quantum numbers corresponding to J_z.



7. Consider three weakly interaction, identical spin 1 particles

   (a) Suppose the space part of the state vector is known to be symmetric under interchange of *any* pair. Using notation $|+\rangle|0\rangle|+\rangle$ for particle 1 in m_s=+1, particle 2 in m_s=0, particle 3 in m_s=+1, and so on, construct the normalized spin states in the following three cases:

      i. All three of them in $|+\rangle$.
      ii. *Two* of them in $|+\rangle$, one in $|0\rangle$.
      iii. All three in different spin states

      What is the total spin in each case?

   (b) Attempt to do the same problem when the space part is antisymmetric under interchange of any pair.

8. Suppose the electron were a spin $\frac{3}{2}$ particle obeying Fermi-Dirac statistics. Write the configuration of a hypothetical Ne (Z=10) atom made up of such "electrons" [that is, the analog of $(1s)^2(2s)^2(2p)^6$]. Show that the configuration is highly degenerate. What is the ground state (the lowest term) of the hypothetical Ne atom in spectroscopic notation ($^{2S+1}L_J$, where S, L, and J stand for the total spin, the total orbital angular momentum, and the total angular momentum, respectively) when exchange splitting and spin-orbit splitting are taken into account?