Solutions:
Homework Set 5 — Scattering 2

Due May 29

1. Without assuming a local potential, the general form of the Lippmann-Schwinger equation is

$$\Psi^+(\mathbf{r}) = \phi(\mathbf{r}) - \frac{2m}{\hbar^2}\int d^3r' d^3r'' G^+_{k'}(\mathbf{r},\mathbf{r}')\langle \mathbf{r}'|V|\mathbf{r}''\rangle\langle \mathbf{r}''|\Psi^+\rangle \tag{1}$$

$$= \phi(\mathbf{r}) - \frac{2m}{\hbar^2}\int d^3r' d^3r'' \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\lambda v(r')v(\mathbf{r}'')\Psi^+(\mathbf{r}'') \tag{2}$$

$$= \phi(\mathbf{r}) - \frac{\lambda}{4\pi}\frac{2m}{\hbar^2}\int d^3r' \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}v(r')\int d^3r'' v(\mathbf{r}'')\Psi^+(\mathbf{r}'') \tag{3}$$

We obtain the Born series by iteration

$$\Psi^+ = \phi(\mathbf{r}) - \frac{2m\lambda}{\hbar^2}\int d^3r' G^+_{k'}(\mathbf{r},\mathbf{r}')v(r')\int d^3r'' v(\mathbf{r}'')\Big[\phi(\mathbf{r}'') - \frac{2m\lambda}{\hbar^2}\int d^3r''' G^+_{k'}(\mathbf{r}',\mathbf{r}'')v(r'')\int d^3r'''' v(\mathbf{r}'''')\Psi^+(\mathbf{r}'''')\Big] \tag{4}$$

At every iteration, the common factors that appear are:

$$B \equiv \int d^3r' v(\mathbf{r}')\phi(\mathbf{r}') = \text{ (independent of } \mathbf{r}) \tag{5}$$

$$A_{\mathbf{k}'}(\mathbf{r}') \equiv \frac{2m}{\hbar^2}\int d^3r'' G^+_{k'}(\mathbf{r}',\mathbf{r}'')v(\mathbf{r}'') \tag{6}$$

So we can rewrite Eq. (4) as

$$\Psi^+(\mathbf{r}) = \phi(\mathbf{r}) - \lambda A_{\mathbf{k}'}(\mathbf{r})\left[B - \lambda\int d^3r'' A_{\mathbf{k}'}(\mathbf{r}'')\left(B - \lambda\int d^3r''' A_{\mathbf{k}'}(\mathbf{r}''')\Big\{\dots\Big\}\right)\right] \tag{7}$$

$$= \phi(\mathbf{r}) - \lambda A_{\mathbf{k}'}(\mathbf{r})B + \lambda^2 A_{\mathbf{k}'}(\mathbf{r})\int d^3r'' A_{\mathbf{k}'}(\mathbf{r}'')B - \lambda^3\dots \tag{8}$$

The first Born approximation is then:

$$\Psi^+(\mathbf{r}) = \phi(\mathbf{r}) - \lambda A_{\mathbf{k}'}(\mathbf{r})B \tag{9}$$

$$= \phi(\mathbf{r}) - \lambda\frac{2m}{\hbar^2}\int d^3r'\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}v(\mathbf{r}')\int d^3r''v(\mathbf{r}')e^{i\mathbf{k}\cdot\mathbf{r}''} \tag{10}$$

$$\underset{r\to\infty}{\to} \phi(\mathbf{r}) - \frac{e^{ikr}}{r}\lambda\frac{m}{2\pi\hbar^2}\int d^3r'e^{i\mathbf{k}'\cdot\mathbf{r}'}v(\mathbf{r}')\int d^3r''e^{i\mathbf{k}\cdot\mathbf{r}''}v(\mathbf{r}'') \tag{11}$$

$$\implies f^{(1)}(\mathbf{k},\mathbf{k}') = -\frac{\lambda m}{2\pi\hbar^2}F_{\mathbf{k}}(v)F_{\mathbf{k}'}(v) \tag{12}$$

where F is the Fourier transform of v.

$$F_{\mathbf{k}}(v) = \int d^3re^{i\mathbf{k}\cdot\mathbf{r}}v(\mathbf{r}). \tag{13}$$

This is in contrast to the case of separable potential, which is proportional to the Fourier transform with respect to the momentum transfer $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$: $F_{\mathbf{q}}(V)$

2. (a) Let $R(r)$ be the radial part of wave function, and

$$\chi(r) = rR(r) \tag{14}$$

with boundary condition

$$\chi(r \to 0) \to 0 \tag{15}$$

χ satisfies

$$\left(\frac{d^2}{dr^2} + k^2 - k_0^2\right)\chi(r) = 0 \tag{16}$$

with

$$k_0^2 = \frac{2mV_0}{\hbar^2}\theta(a-r) \tag{17}$$

For $r > a$, χ is solution to free equation. The s-wave is a linear combination of $\sin(kr)$ and $\cos(kr)$, or equivalently $\sin(kr + \delta_0)$.

Similarly, for $r < a$, χ is a linear combination of $\sinh(\alpha r)$ and $\cosh(\alpha r)$ with $\alpha^2 = k_0^2 - k^2 > 0$ (note that $E < V_0$).

The boundary condition allows only the cosh term.

$$\chi(r) = \begin{cases} A\sinh(\alpha r) & r < a \\ B\sin(kr + \delta_0) & r > a \end{cases} \tag{18}$$

At $r = a$, match χ and χ':

$$A\sinh(\alpha a) = B\sin(ka + \delta_0) \tag{19}$$

$$\alpha A\cosh(\alpha a) = kB\cos(ka + \delta_0) \tag{20}$$

$$\implies k\tanh(\alpha a) = \alpha\tan(ka + \delta_0) \tag{21}$$

$$\delta_0(k) = -ka + \tan^{-1}\left[\frac{k}{\sqrt{k_0^2 - k^2}}\tanh\left(a\sqrt{k_0^2 - k^2}\right)\right] \tag{22}$$

(b) For $V \to \infty$, $\tanh \to 1 \implies \delta_0 \to -ka$

(c)

$$\lim_{k\to 0}\delta = -k\alpha \tag{23}$$

$$\alpha = a - \frac{1}{k_0}\tanh(k_0 a) \tag{24}$$

$$\sigma = \lim_{k\to 0}\left[\frac{4\pi\sin^2\delta}{k^2}\right] \tag{25}$$

$$= 4\pi\lim_{k\to 0}\left[\frac{\delta}{k}\right]^2 \tag{26}$$

$$= 4\pi\alpha^2 \tag{27}$$

$$= 4\pi\left[a - \frac{1}{k_0}\tanh(k_0 a)\right]^2 \tag{28}$$

3. In the limit of low energy, only the s-wave contributes, $l = 0$:

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{d^2}{dr^2}r - \cancel{\frac{l(l+1)}{r^2}}\right) + V(r) - \frac{\hbar^2 k^2}{2m}\right]R_0(r) = 0 \tag{29}$$

Let $\chi(r) \equiv rR(r)$:

$$\left[\frac{d^2}{dr^2} + \gamma\delta(r-a) + k^2\right]\chi(r) = 0 \tag{30}$$

with $\gamma \equiv \frac{2mC}{\hbar^2}$. The general solution inside (regular at the origin) and outside the shell is:

$$\chi = \begin{cases} A\sin(kr) & r < a \\ B\sin(kr+\delta) & r < a \end{cases} \tag{31}$$

χ should be continuous at a, and χ' should have the correct discontinuity for the δ function:

$$A\sin(ka) = B\sin(ka+\delta) \tag{32}$$

$$0 = \left.\frac{d\chi}{dr}\right|_{a^+} - \left.\frac{d\chi}{dr}\right|_{a^-} + \gamma\chi(a) \tag{33}$$

$$= kB\cos(ka+\delta) - Ak\cos(ka) + \gamma A\sin(ka) \tag{34}$$

$$\implies \tan(ka+\delta) = \frac{\tan(ka)}{1-\frac{\gamma}{k}\tan(ka)} \tag{35}$$

with $\gamma = \frac{2mC}{\hbar^2}$.

For low energy, $k \to 0$:

$$\delta = \tan^{-1}\left[\frac{\tan(ka)}{1-\frac{\gamma}{k}\tan(ka)}\right] - ka \tag{36}$$

$$\simeq \frac{a^2\gamma k}{1-\gamma a} + O(k^3) \tag{37}$$

$$\sigma \simeq \frac{4\pi}{k^2}\delta^2 \tag{38}$$

$$= \frac{4\pi\gamma^2 a^2}{(1-\gamma a)^2} \tag{39}$$

4. (a) In the Born approximation:

$$f^{(1)}(\mathbf{k}',\mathbf{k}) = \frac{-m}{2\pi\hbar^2}\int d^3x' e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}'}V(\mathbf{x}') \tag{40}$$

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\mathbf{k}',\mathbf{k})|^2 \tag{41}$$

$$= \frac{m^2}{4\pi^2\hbar^4}\int d^3x' d^3x'' e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}'}V(\mathbf{x}')e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}''}V(\mathbf{x}'') \tag{42}$$

$$= \frac{m^2}{4\pi^2\hbar^4}\int d^3x' d^3x'' V(\mathbf{x}')V(\mathbf{x}'')e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}e^{-i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x}'')} \tag{43}$$

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} \tag{44}$$

$$= \frac{m^2}{4\pi^2\hbar^4}\int d^3x' d^3x'' V(\mathbf{x}')V(\mathbf{x}'')e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}\int d\Omega_{k'} e^{-i\mathbf{k}'\cdot(\mathbf{x}'-\mathbf{x}'')} \tag{45}$$

$$= \frac{m^2}{4\pi^2\hbar^4}\int d^3x' d^3x'' V(\mathbf{x}')V(\mathbf{x}'')e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}2\pi\int_{-1}^{1} d\cos\theta e^{ik'|\mathbf{x}'-\mathbf{x}''|\cos\theta} \tag{46}$$

$$= \frac{m^2}{4\pi^2\hbar^4}\int d^3x' d^3x'' V(\mathbf{x}')V(\mathbf{x}'')e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}4\pi\frac{\sin(k|\mathbf{x}'-\mathbf{x}''|)}{k|\mathbf{x}'-\mathbf{x}''|} \tag{47}$$

(Recall $k' = k$ from energy conservation in elastic scattering).

In this case $V(\mathbf{x}) = V(r)$, and the cross section cannot depend on the direction of $\mathbf{k}$ by symmetry $\implies$ we can average the expression over angle and replace

$$e^{i(\mathbf{k})\cdot(\mathbf{x}'-\mathbf{x}'')} \to \left\langle e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}\right\rangle \tag{48}$$

$$= \frac{1}{2}\int_{-1}^{1} d\cos\theta \; e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')} \tag{49}$$

$$= \frac{\sin(k|\mathbf{x}'-\mathbf{x}''|)}{k|\mathbf{x}'-\mathbf{x}''|} \tag{50}$$

$$\implies \sigma = \frac{m^2}{\pi\hbar^4}\int d^3x' d^3x'' V(r')V(r'')\frac{\sin^2(k|\mathbf{x}'-\mathbf{x}''|)}{k^2|\mathbf{x}'-\mathbf{x}''|^2} \tag{51}$$

(b) The optical theorem in the second Born approximation reads:

$$\sigma = \frac{4\pi}{k}\Im\left[f^{(2)}(\mathbf{k}=\mathbf{k}')\right] \tag{52}$$

$$= \Im\left[\frac{4\pi}{k}\left(-\frac{m}{2\pi\hbar^2}\right)^2\int d^3x' d^3x'' e^{-i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}'')}\frac{e^{ik|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}'-\mathbf{x}''|}V(r')V(r'')\right] \tag{53}$$

Again we can perform the angular integral explicitly

$$\sigma = \Im\left[-\frac{m^2}{\pi\hbar^4 k}\int d^3x' d^3x'' \frac{\sin(k|\mathbf{x}'-\mathbf{x}''|)}{k|\mathbf{x}'-\mathbf{x}''|}\frac{e^{ik|\mathbf{x}'-\mathbf{x}''|}}{|\mathbf{x}'-\mathbf{x}''|}V(r')V(r'')\right] \tag{54}$$

$$= \frac{m^2}{\pi\hbar^4}\int d^3x' d^3x'' V(r')V(r'')\frac{\sin^2(k|\mathbf{x}'-\mathbf{x}''|)}{k^2|\mathbf{x}'-\mathbf{x}''|^2} \tag{55}$$

5. (a) In the Born approximation:

$$f^{(1)}(\mathbf{k}',\mathbf{k}) = \frac{-m}{2\pi\hbar^2}\int d^3x' e^{-i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}') \tag{56}$$

$$= \frac{-mV_0}{2\pi\hbar^2}\int d^3x' e^{-i\mathbf{q}\cdot\mathbf{r}'}\left[\delta^{(3)}(\mathbf{r}-\epsilon\hat{z}) - \delta^{(3)}(\mathbf{r}+\epsilon\hat{z})\right] \tag{57}$$

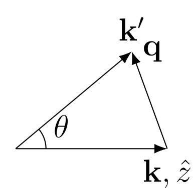
$$= \frac{-mV_0}{2\pi\hbar^2}\left(e^{i\epsilon\mathbf{q}\cdot\hat{z}} - e^{-i\epsilon\mathbf{q}\cdot\hat{z}}\right) \tag{58}$$

$$= \frac{-mV_0}{2\pi\hbar^2} i\sin(\epsilon q_z) \tag{59}$$

with

$$q_z \equiv (\mathbf{k}'-\mathbf{k})_z \tag{60}$$

$$= -2k\sin^2\frac{\theta}{2} \tag{61}$$



$$\frac{d\sigma}{d\Omega} = |f|^2 \tag{62}$$

$$= \frac{m^2V_0^2}{4\pi^2\hbar^4}\sin^2(-2k\epsilon\sin^2\frac{\theta}{2}) \tag{63}$$

(b) We expect the Born approximation to be good when

$$1 \gg \left| \frac{2m}{\hbar^2} \int d^3x' \frac{e^{ik|x'|}}{4\pi|x'|} V(x') e^{i\vec{k}\cdot\vec{x}'} \right| \tag{64}$$

$$= \left| \frac{2mV_0}{\hbar^2} \frac{e^{ik|\epsilon|}}{4\pi|\epsilon|} \left[e^{i\epsilon k_z} - e^{-i\epsilon k_z} \right] \right| \tag{65}$$

$$= \frac{m|V_0|}{\pi\hbar^2\epsilon} |\sin(\epsilon k)| \tag{66}$$

$$\Longrightarrow |V_0| \ll \frac{\pi\epsilon\hbar^2}{m|\sin(\epsilon k)|} \tag{67}$$

6. (a) The potential energy $V(\mathbf{r})$ is related to the charge distribution by

$$\nabla^2 V(r) = -4\pi e \rho(r) \tag{68}$$

(in Gaussian units.) Taking the Fourier transform, this reads

$$\tilde{V}(q) = \frac{4\pi e}{q^2} \tilde{\rho}(q) \tag{69}$$

In the Born approximation, the scattering amplitude is proportional to the Fourier transform of V

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = \frac{-m}{2\pi\hbar^2} \tilde{V}(q) \tag{70}$$

$$= \frac{-2me}{\hbar^2} \frac{\tilde{\rho}(q)}{q^2} \tag{71}$$

with $q = |\mathbf{k}' - \mathbf{k}| = 2k \sin\frac{\theta}{2}$.

$$\frac{d\sigma}{d\Omega} = |f|^2 \tag{72}$$

$$= \frac{4e^2m^2}{\hbar^4} \frac{\tilde{\rho}^2(2k\sin\frac{\theta}{2})}{(2k\sin\frac{\theta}{2})^4} \tag{73}$$

(b) For $\theta \to 0$, $q \to 0$.

$$\frac{d\sigma}{d\Omega}(\theta = 0) = \lim_{q \to 0} \frac{4e^2m^2}{\hbar^4} \frac{\tilde{\rho}^2(q)}{q^4} \tag{74}$$

Expanding for small q:

$$\tilde{\rho}(q) = \int d^3x\rho(r)e^{i\mathbf{q}\cdot\mathbf{x}} \tag{75}$$

$$= \int d^3x\rho(r)\left[1 - i\mathbf{q}\cdot\mathbf{x} - \frac{(\mathbf{q}\cdot\mathbf{x})^2}{2} + \dots\right] \tag{76}$$

$$= \cancel{\int d^3x\rho(r)} - i\cancel{\int d^3xqr\cos\theta\rho(r)} - \frac{q^2}{2}\int d^3xr^2\cos^2\theta\rho(r) + \dots \tag{77}$$

where θ is chosen to be the angle with respect to $\mathbf{q}$. The first term is zero because the net charge is zero, and the second term is zero from rotational symmetry.

We can perform the integral over θ:

$$\int_{-1}^{1} d(\cos\theta)\cos^2\theta = \frac{2}{3} \tag{78}$$

$$= \int_{-1}^{1} d(\cos\theta)\frac{1}{3} \tag{79}$$

$$\implies \lim_{q\to 0}\tilde{\rho}(q) = -\frac{q^2}{6}\int d^3xr^2\rho(r) \tag{80}$$

$$= -\frac{q^2A}{6} \tag{81}$$

$$\frac{d\sigma}{d\Omega}(\theta = 0) = \frac{e^2m^2A^2}{9\hbar^4} \tag{82}$$

(c) A hydrogen atom consists of a nucleus at the center, and an electron, whose charge distribution is given by the square of the ground state wave function $\Psi_{1s}^2(r)$:

$$\rho(r) = -e\left[\delta^3(\mathbf{x}) - \Psi_{1s}^2(r)\right] \tag{83}$$

$$\Psi_{1s}(r) = \frac{2e^{-r/a_0}}{\sqrt{a_0^3}} \tag{84}$$

$$\tilde{\rho}(q) = -e\int d^3x \left[\delta^3(\mathbf{x}) - \frac{4}{a_0^3}e^{-2r/a_0}\right] e^{i\mathbf{q}\cdot\mathbf{x}} \tag{85}$$

$$= -e\left[1 - \frac{16}{(4+q^2a_0^2)^2}\right] \tag{86}$$

$$= -e\frac{q^2a_0^2}{2} + O(q^4) \tag{87}$$

$$A = -6\lim_{q\to 0}\frac{\tilde{\rho}(q)}{q^2} \tag{88}$$

$$= 3ea_0^2 \tag{89}$$

$$\frac{d\sigma}{d\Omega}(\theta \to 0) = a_0^2 \tag{90}$$

7. Consider the (non-relativistic) scattering of an electron of mass m and momentum k through an angle θ. Calculate the differential cross-section $\frac{d\sigma}{d\Omega}$ in the Born approximation for the spin-dependent potential

$$V = e^{-\mu r^2}[A + B\vec{\sigma}\cdot\mathbf{r}], \tag{91}$$

where $\vec{\sigma}$ are the Pauli matrices and μ, A, B are constants. Assume that the initial spin is polarized along the incident direction and sum over all final spins.

The Born approximation has the form

$$f^{(1)} = \frac{m(2\pi)^2}{\hbar^2}\langle\mathbf{k}', m_s'|V|\mathbf{k}, m_s\rangle \tag{92}$$

The initial spin is polarized, $m_s = +1/2$, and we want the cross section summed over final spins:

$$\frac{d\sigma}{d\Omega} = \frac{m^2(2\pi)^4}{\hbar^4}\left[|\langle\mathbf{k}', +|V|\mathbf{k}, +\rangle|^2 + |\langle\mathbf{k}', -|V|\mathbf{k}, +\rangle|^2\right] \tag{93}$$

$$= \frac{m^2}{4\pi^2\hbar^4}\left[|\langle +|\tilde{V}(q)|+\rangle|^2 + |\langle -|\tilde{V}(q)|+\rangle|^2\right] \tag{94}$$

The spatial part is proportional to the Fourier transform, as usual, and we are interested only in the spin transitions $+ \to +$ and $+ \to -$

$$\tilde{V}(\mathbf{q}) = \langle \mathbf{k}'|V|\mathbf{k}\rangle \tag{95}$$

$$= \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) \tag{96}$$

$$= \frac{p^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \left[A + \frac{i}{2\mu} B\mathbf{q}\cdot\tilde{\sigma} \right], \tag{97}$$

(which is a 2x2 matrix)

Choose the incoming wave in the z direction, and write the transverse components in the basis

$$\sigma^{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y) \tag{98}$$

$$q^{\pm} = q_x \pm iq_y \tag{99}$$

$$\implies \tilde{V}(\mathbf{q}) = \frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \left[A + \frac{i}{2\mu} B(q_z\sigma_z + q^+\sigma^- + q^-\sigma^+) \right] \tag{100}$$

The transition $+ \to +$ only has contribution from the diagonal terms (A and the term with σ_z). The transition $+ \to -$ only has contribution from the lowering operator σ^-. The raising operator σ^+ has no affect when the incoming spin is $+$:

$$\langle +|\sigma_z|+\rangle = 1 \tag{101}$$

$$\langle -|\sigma^-|+\rangle = 1 \tag{102}$$

$$\implies \frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} \left[\frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \right]^2 \left[|A + \frac{iq_z}{2\mu}B|^2 + |\frac{iq^+B}{2\mu}|^2 \right] \tag{103}$$

$$= \frac{m^2}{4\pi^2\hbar^4} \left[\frac{\pi^{3/2}}{\mu^{3/2}} e^{-q^2/4\mu} \right]^2 \left[A^2 + \frac{B^2q^2}{4\mu^2} \right] \tag{104}$$

$$= \frac{m\pi}{4\hbar^4\mu^3} e^{-q^2/2\mu} \left[A^2 + \frac{B^2q^2}{4\mu^2} \right] \tag{105}$$

$$= \frac{m\pi}{4\hbar^4\mu^3} e^{-k^2(1-\cos\theta)/\mu} \left[A^2 + \frac{B^2k^2(1-\cos\theta)}{2\mu^2} \right] \tag{106}$$