

Solutions:

Homework Set 4 — Scattering 1

Due May 15

1. At 100 m radius, 10 m^2 covers

$$\frac{4\pi 10\text{m}^2}{4\pi(100\text{m})^2} = 10^{-3} \text{ steradians} = \Omega. \quad (1)$$

$d\sigma/d\Omega$ is approximately constant over this solid angle (compared, e.g., to our one significant figure knowledge of the incoming flux),

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \left. \frac{d\sigma}{d\Omega} \right|_{\theta=30^\circ} \int d\Omega \quad (2)$$

$$= 10^{-34} \sin^2(30^\circ) \text{cm}^2 \cdot 10^{-3} \quad (3)$$

$$= \frac{1}{4} \times 10^{-37} \text{cm}^2 = 2.5 \times 10^{-38} \text{cm}^2 \quad (4)$$

The total number of particles is this times the incoming flux:

$$\frac{2 \times 10^{32} \cdot 2.5 \times 10^{-38} \text{ particles}}{\text{unit time}} = 5 \times 10^{-6} \text{ particles/time} \quad (5)$$

2. (a) The $E \rightarrow E + i\epsilon$ prescription is correct:

$$G_+(x, x') = \frac{\hbar^2}{2m} \langle x | \frac{1}{E - H_0 + i\epsilon} | x' \rangle \quad (6)$$

$$= \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \langle x | p \rangle \langle p | \frac{1}{E - H_0 + i\epsilon} | p' \rangle \langle p' | x' \rangle \quad (7)$$

$$= \frac{\hbar^2}{2m} \frac{1}{2\pi\hbar} \int dp dp' e^{ipx/\hbar} \frac{\delta(p - p')}{E - H_0 + i\epsilon} e^{-ip'x'/\hbar} \quad (8)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' \frac{e^{ik'(x-x')}}{k^2 - k'^2 + i\epsilon} \quad (p' = \hbar k', E = \frac{\hbar^2 k^2}{2m}) \quad (9)$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} dk' \frac{e^{ik'(x-x')}}{[k' + (k + i\epsilon)][k' - (k + i\epsilon)]} \quad (10)$$

$$= -\frac{1}{2\pi} \begin{cases} 2\pi i \frac{1}{2k} e^{ik(x-x')}, & x > x' \text{ (close contour in upper half plane)} \\ -2\pi i \frac{1}{2k} e^{-ik(x-x')}, & x < x' \text{ (close contour in lower half plane)} \end{cases} \quad (11)$$

$$= -\frac{i}{2k} e^{ik|x-x'|} \quad (12)$$

$$\langle x | \Psi^+ \rangle = \langle x | \phi \rangle - \frac{2m}{\hbar^2} \int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|} V(x') \langle x' | \Psi^+ \rangle \quad (\text{assuming local } V) \quad (13)$$

(b)

$$V = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x), \quad (\gamma > 0). \quad (14)$$

$$\implies \langle x|\Psi^+\rangle = \langle x|\phi\rangle - \frac{2m}{\hbar^2} \int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|} V(x') \langle x'|\Psi^+\rangle \quad (15)$$

$$= \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{2m}{\hbar^2} \int_{-\infty}^{\infty} dx' \frac{i}{2k} e^{ik|x-x'|} \left(\frac{\gamma\hbar^2}{2m}\right) \delta(x') \langle x'|\Psi^+\rangle \quad (16)$$

$$= \frac{e^{ikx}}{\sqrt{2\pi}} + \frac{i\gamma}{2k} e^{ik|x|} \langle 0|\Psi^+\rangle \quad (17)$$

$$\langle 0|\Psi^+\rangle = \frac{1}{\sqrt{2\pi}} + \frac{i\gamma}{2k} \langle 0|\Psi^+\rangle \quad (18)$$

$$\implies \langle 0|\Psi^+\rangle = \frac{1}{\sqrt{2\pi}} \left(1 - \frac{i\gamma}{2k}\right)^{-1} \quad (19)$$

$$\implies \langle x|\Psi^+\rangle = \frac{1}{\sqrt{2\pi}} \left(e^{ikx} + e^{ik|x|} \frac{i\gamma}{2k} \left(1 - \frac{i\gamma}{2k}\right)^{-1} \right) \quad (20)$$

$$= \frac{1}{\sqrt{2\pi}} \begin{cases} \left(1 - \frac{i\gamma}{2k}\right)^{-1} e^{ikx}, & x > 0 \\ e^{ikx} + \left(\frac{2k}{i\gamma} - 1\right)^{-1} e^{-ikx}, & x < 0 \end{cases} \quad (21)$$

$$\implies T = \left(1 - \frac{i\gamma}{2k}\right)^{-1}, \quad (22)$$

$$R = \left(\frac{2k}{i\gamma} - 1\right)^{-1}$$

(c) Analytically continuing the wavefunction to complex k , the poles of T and R are both at $k = i\gamma/2$, and $\langle x|\Psi\rangle \sim e^{-\gamma|x|/2}$.

The Schrodinger equation,

$$-\frac{1}{2m} \frac{d^2\Psi}{dx^2} - \frac{\gamma}{2m} \delta(x)\Psi = -|E|\Psi, \quad (23)$$

has solutions of the form

$$\Psi \sim \begin{cases} e^{-\kappa x} & (x > 0) \\ e^{\kappa x} & (x < 0), \end{cases} \quad (24)$$

with $k = i\kappa = i\sqrt{2m|E|}$, satisfying

$$\frac{d\Psi}{dx} \Big|_{0+} - \frac{d\Psi}{dx} \Big|_{0-} = -\gamma\Psi(0) \quad (25)$$

which implies that $\kappa = \gamma/2$, or $k = i\gamma/2$.

Thus $\Psi \sim e^{-\gamma|x|/2}$, in agreement with the expectation that bound states are represented by poles in the scattering amplitude on the imaginary momentum axis.

3. (a)

$$E = \frac{\hbar^2 k^2}{2m} > V_0 \quad (26)$$

$$\psi(x) = \begin{cases} e^{ikx} + r e^{-ikx} & x < 0 \\ t e^{ik'x} & x > 0 \end{cases} \quad (27)$$

$$k'^2 = k^2 - \frac{2mV_0}{\hbar^2} \quad (28)$$

Match at $x=0$:

$$\begin{cases} 1 + r = t \\ ik(1 - r) = ik't \end{cases} \quad (29)$$

$$r = \frac{k - k'}{k + k'} \quad (30)$$

$$t = \frac{2k}{k + k'} \quad (31)$$

$$R = |r|^2 = \frac{(k - k')^2}{(k + k')^2} \quad (32)$$

$$T = |t|^2 = \frac{4k^2}{(k + k')^2} \quad (33)$$

(b)

$$J_R = \frac{\hbar k}{m} R = \frac{\hbar k(k - k')^2}{m(k + k')^2} \quad (34)$$

$$J_T = \frac{\hbar k'}{m} T = \frac{\hbar 4k'k^2}{m(k + k')^2} \quad (35)$$

$$J_R + J_T = \frac{\hbar k}{m} \quad (36)$$

4. (a) The quantity to be calculated,

$$\frac{\hbar^2}{2m} \langle \mathbf{x} | \frac{1}{E - H_0 + i\epsilon} | \mathbf{x}' \rangle, \quad (37)$$

is the Green's function for (-) the Helmholtz operator $(\nabla^2 + k^2)$:

$$(\nabla^2 + k^2) \frac{\hbar^2}{2m} \langle \mathbf{x} | \frac{1}{E - H_0 + i\epsilon} | \mathbf{x}' \rangle = (\nabla^2 + k^2) G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x}, \mathbf{x}'). \quad (38)$$

$[\nabla^2 + k^2, L^2] = 0$, so they can be simultaneously diagonalized, and we look for a solution of the form:

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l,m} g_{lm}(r, r'; k) Y_{lm}(\hat{r}) Y_{lm}(\hat{r}') \quad (39)$$

$$\implies (\nabla^2 + k^2) G(\mathbf{x}, \mathbf{x}') = \sum_{l,m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + k^2 \right) g_{lm} Y_{lm}(\hat{r}) Y_{lm}(\hat{r}') \quad (40)$$

$$= \delta^3(\mathbf{x} - \mathbf{x}') = \frac{1}{r^2} \delta(r - r') \sum_{l,m} Y_{lm}(\hat{r}) Y_{lm}(\hat{r}') \quad (41)$$

$$\implies \frac{1}{r^2} \delta(r - r') = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + k^2 \right) g_l(r, r'; k). \quad (42)$$

We find general solutions for the homogeneous equation at $r \neq r'$, and then match the two solutions at $r = r'$.

The general homogeneous solution is

$$g_l = C_1 j_l(kr) = C_2 n_l(kr), \quad (43)$$

where j and n are the spherical Bessel and Neumann functions.

The Helmholtz operator is Hermitian, which means g_l is also, so it must be symmetric under interchange of r and r' :

$$g_l = C_l g_{<}(r_{<}) g_{>}(r_{>}) \quad (44)$$

It must be regular at the origin, so

$$g_{<}(r_{<}) = j_l(kr_{<}). \quad (45)$$

We want only outgoing waves at $r \rightarrow \infty$, so

$$g_{>}(r_{>}) = j_l(kr_{>}) + i n_l(kr_{>}) \equiv h_l^{(1)}(kr_{>}) \quad (46)$$

Matching derivatives at ($r = r'$):

$$\lim_{\epsilon \rightarrow 0} g'_l|_{r'-\epsilon}^{r'+\epsilon} = \frac{1}{r^2} \quad (47)$$

$$= C_l j_l(jr') h_l^{(1)}(kr)|_{r'} - C_l j'_l(kr)|_{r'} h_l^{(1)}(kr') \quad (48)$$

$$= C_l W(j_l, h_l^{(1)}) \quad (49)$$

where W is the wronskian.

$$\implies C_l \frac{i}{kr^2} = \frac{1}{r^2} \quad (50)$$

Putting everything together:

$$\frac{\hbar^2}{2m} \langle \mathbf{x} | \frac{1}{E - H_0 + i\epsilon} | \mathbf{x}' \rangle = -ik \sum_l \sum_m Y_{lm}(\hat{r}) Y_{lm}^*(\hat{r}') j_l(kr_{<}) h_l^{(1)}(kr_{>}) \quad (51)$$

(b)

$$\langle \mathbf{x} | Elm(+) \rangle = \langle \mathbf{x} | Elm \rangle + \langle \mathbf{x} | \frac{1}{E - H_0 + i\epsilon} V | Elm(+) \rangle \quad (52)$$

$$= \langle \mathbf{x} | Elm \rangle + \frac{2m}{\hbar^2} \int d^3 x' G_+(\mathbf{x}, \mathbf{x}') V(r') \langle \mathbf{x}' | Elm(+) \rangle \quad (53)$$

$$= \langle \mathbf{x} | Elm \rangle - \frac{2m}{\hbar^2} \int d^3 x' ik \sum_{l,m} Y_l^m(\hat{r}) Y_l^{m*}(\hat{r}') j_l(kr_{<}) h_l^{(1)}(kr_{>}) V(r') \Psi_{Elm}^+(\mathbf{x}') \quad (54)$$

The free state is

$$\langle \mathbf{x} | Elm \rangle = C_l j_l(kr) Y_l^m(\hat{r}), \quad (55)$$

while a general scattering state can be written as

$$\langle \mathbf{x} | Elm(+) \rangle = C_l A(k, r) Y_l^m(\hat{r}). \quad (56)$$

Pluggin in:

$$C_l A(k, r) Y_l^m(\hat{r}) = C_l j_l(kr) Y_l^m(\hat{r}) \quad (57)$$

$$- \frac{2mik}{\hbar^2} \int d^3 x' \sum_{l', m'} Y_{l'}^{m'}(\hat{r}) Y_{l'}^{m'*}(\hat{r}') j_{l'}(kr_{<}) h_{l'}^{(1)}(kr_{>}) V(r') C_l A(k, r') Y_l^m(\hat{r}') \quad (58)$$

$$= C_l j_l(kr) Y_l^m(\hat{r}) - \frac{2mik}{\hbar^2} \int dr' r'^2 j_{l'}(kr_{<}) h_{l'}^{(1)}(kr_{>}) V(r') C_l A(k, r') Y_l^m(\hat{r}') \quad (59)$$

$$\implies A_l(k; r) = j_l(kr) - \frac{2mik}{\hbar^2} \int_0^\infty j_l(kr_{<}) h_l^{(1)}(kr_{>}) V(r') A_l(k; r') r'^2 dr' \quad (60)$$

For $r \rightarrow \infty$, then $r_{<} = r'$, $r_{>} = r$,

$$A_l(r \rightarrow \infty) = j_l(kr) - \frac{2mik}{\hbar^2} h_l^{(1)}(kr) \int_0^\infty j_l(kr') V(r') A_l(k; r') r'^2 dr' \quad (61)$$

$$= e^{i\delta_l} [\cos \delta_l j_l(kr) - \sin \delta_l h_l(kr)] \quad (62)$$

$$\implies f_l(k) = e^{i\delta_l} \frac{\sin \delta_l}{k} \quad (63)$$

$$= - \left(\frac{2m}{\hbar^2} \right) \int_0^\infty j_l(kr) A_l(k; r) V(r) r^2 dr \quad (64)$$