

# Homework Set 3 — Symmetries 2

Due April 17

1. (a) The plane wave is

$$\psi(\mathbf{x}, t) = e^{i(\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} - \omega t)} \quad (1)$$

Reversing the direction of momentum  $\mathbf{p} \rightarrow -\mathbf{p}$  gives

$$\psi(\mathbf{x}, t) \rightarrow e^{i(-\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} - \omega t)} \quad (2)$$

$$= e^{-i(\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} + \omega t)} \quad (3)$$

$$= \psi^*(\mathbf{x}, -t) \quad (4)$$

- (b) From Sakurai (3.2.52):

$$\chi(\hat{\mathbf{n}}) = \begin{pmatrix} \cos \frac{\beta}{2} e^{-i\alpha/2} \\ \sin \frac{\beta}{2} e^{i\alpha/2} \end{pmatrix}, \quad (5)$$

Note that this is defined so that

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \chi = \chi. \quad (6)$$

It has spin in the direction of  $\hat{\mathbf{n}}$ . (E.g., the usual spin up state for  $\hat{\mathbf{n}} = \hat{z}$ ).

We then have

$$\chi^*(\hat{\mathbf{n}}) = \begin{pmatrix} \cos \frac{\beta}{2} e^{i\alpha/2} \\ \sin \frac{\beta}{2} e^{-i\alpha/2} \end{pmatrix}, \quad (7)$$

$$-i\sigma_2 \chi^*(\hat{\mathbf{n}}) = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\beta}{2} e^{i\alpha/2} \\ \sin \frac{\beta}{2} e^{-i\alpha/2} \end{pmatrix}, \quad (8)$$

$$= \begin{pmatrix} -\sin \frac{\beta}{2} e^{-i\alpha/2} \\ \cos \frac{\beta}{2} e^{i\alpha/2} \end{pmatrix} \quad (9)$$

But this has opposite eigenvalue:

$$\sigma \cdot \hat{\mathbf{n}}[-i\sigma_2\chi^*(\hat{\mathbf{n}})] = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix} \begin{pmatrix} -\sin \frac{\beta}{2} e^{-i\alpha/2} \\ \cos \frac{\beta}{2} e^{i\alpha/2} \end{pmatrix} \quad (10)$$

$$= \begin{pmatrix} \cos \beta & \sin \beta \cos \alpha - i \sin \beta \sin \alpha \\ \sin \beta \cos \alpha + i \sin \beta \sin \alpha & -\cos \beta \end{pmatrix} \begin{pmatrix} -\sin \frac{\beta}{2} e^{-i\alpha/2} \\ \cos \frac{\beta}{2} e^{i\alpha/2} \end{pmatrix} \quad (11)$$

$$= - \begin{pmatrix} -\sin \frac{\beta}{2} e^{-i\alpha/2} \\ \cos \frac{\beta}{2} e^{i\alpha/2} \end{pmatrix} \quad (12)$$

$$= -[-i\sigma_2\chi^*(\hat{\mathbf{n}})] \quad (13)$$

$\implies$  it is the two-component eigenspinor with spin direction reversed. (E.g., spin down for  $\hat{\mathbf{n}} = \hat{z}$ , or spin up for  $\hat{\mathbf{n}} = -\hat{z}$ )

2. (a) Time reversal commutes with the rotation operator:

$$\Theta \mathcal{D} \Theta^{-1} = \Theta e^{-iJ \cdot \hat{\mathbf{n}} \theta / \hbar} \Theta^{-1} \quad (14)$$

$$= e^{-(\Theta i J \Theta^{-1}) \cdot \hat{\mathbf{n}} \theta / \hbar} \quad (15)$$

$$= e^{-(\Theta i \Theta^{-1} \Theta J \Theta^{-1}) \cdot \hat{\mathbf{n}} \theta / \hbar} \quad (16)$$

$$= e^{-iJ \cdot \hat{\mathbf{n}} \theta / \hbar} = \mathcal{D} \quad (17)$$

since  $\Theta i \Theta^{-1} = -1$  and  $\Theta J \Theta^{-1} = -J$ .

$$\implies \Theta \mathcal{D} = \mathcal{D} \Theta$$

So we have

$$\Theta \mathcal{D}(R) |j, m\rangle = \mathcal{D}(R) \Theta |j, m\rangle \quad (18)$$

$$= i^{2m} \mathcal{D}(R) |j, -m\rangle \quad (19)$$

(b) Consider

$$\langle j, -m' | \Theta \mathcal{D}(R) |j, -m\rangle = i^{-2m} \langle j, -m' | \mathcal{D}(R) |j, -m\rangle \quad (20)$$

$$= i^{-2m} \mathcal{D}_{-m' -m}^{(j)}(R) \quad (21)$$

(see part 2a).

But we also have

$$\langle j, -m' | \Theta \mathcal{D}(R) | j, -m \rangle = \sum_{m''} \langle j, -m' | \Theta \left[ |j, m''\rangle \langle j, m'' | \mathcal{D}(R) | j, m \rangle \right], \quad (22)$$

$$= \sum_{m''} \left[ \langle j, -m' | \Theta | j, m'' \rangle \right] \langle j, m'' | \mathcal{D}(R) | j, m \rangle^*, \quad (23)$$

$$= i^{2m''} \sum_{m''} \langle j, -m' | j, -m'' \rangle \langle j, m'' | \mathcal{D}(R) | j, m \rangle^* \quad (24)$$

$$= i^{2m''} \sum_{m''} \delta_{m', m''} \langle j, m'' | \mathcal{D}(R) | j, m \rangle^* \quad (25)$$

$$= i^{2m'} \langle j, m' | \mathcal{D}(R) | j, m \rangle^* \quad (26)$$

$$= i^{2m'} \mathcal{D}_{m'm}^{(j)*}(R) \quad (27)$$

(Note that  $\Theta$  acts on everything to the right, and hence the complex conjugation in the second line.)

Comparing the two results, we have:

$$\mathcal{D}_{m'm}^{(j)*}(R) = i^{-2(m'+m)} \mathcal{D}_{-m'-m}^{(j)}(R), \quad (28)$$

3. Since both  $\mathbf{p}^2$  and  $\mathbf{x}$  are invariant under time reversal, so is  $H$ :  $[H, \Theta] = 0$ . If there is no degeneracy, any energy eigenstate  $|\alpha\rangle$  must also be an eigenstate of  $\Theta$ :

$$\Theta |\alpha\rangle = |\tilde{\alpha}\rangle = e^{-\delta} |\alpha\rangle \quad (29)$$

The expectation  $\langle \mathbf{L} \rangle$  is real, since  $\mathbf{L}$  is Hermitian:

$$\langle \mathbf{L} \rangle = \langle \mathbf{L}^\dagger \rangle^* \quad (30)$$

$$= \langle \mathbf{L} \rangle^* \quad (31)$$

If we define  $|\beta\rangle \equiv \mathbf{L}|\alpha\rangle$ , then by the antiunitarity of time reversal:

$$\langle\alpha|\mathbf{L}|\alpha\rangle = \langle\alpha|\beta\rangle \quad (32)$$

$$= \langle\tilde{\alpha}|\tilde{\beta}\rangle^* \quad (33)$$

$$= \langle\tilde{\alpha}|\tilde{\beta}\rangle \quad (34)$$

$$= \langle\tilde{\alpha}|\Theta\mathbf{L}|\alpha\rangle \quad (35)$$

$$= \langle\tilde{\alpha}|\Theta\mathbf{L}\Theta^{-1}\Theta|\alpha\rangle \quad (36)$$

$$= -\langle\tilde{\alpha}|\mathbf{L}\Theta|\alpha\rangle \quad (37)$$

$$= -\langle\tilde{\alpha}|\mathbf{L}|\tilde{\alpha}\rangle \quad (38)$$

$$= -e^{-i\delta}\langle\alpha|\mathbf{L}|\alpha\rangle e^{i\delta} \quad (39)$$

$$= -\langle\alpha|\mathbf{L}|\alpha\rangle \quad (40)$$

$$\implies \langle\mathbf{L}\rangle = 0 \quad (41)$$

If we expand the wavefunction in spherical harmonics:

$$\psi_\alpha(\mathbf{x}) \equiv \langle\mathbf{x}|\alpha\rangle \quad (42)$$

$$= \sum_{l,m} \langle\mathbf{x}|l,m\rangle \langle l,m|\alpha\rangle \quad (43)$$

$$= \sum_{l,m} F_{l,m}(r) Y_l^m(\theta, \phi), \quad (44)$$

then

$$\langle\mathbf{x}|\tilde{\alpha}\rangle = \langle\mathbf{x}|\Theta|\alpha\rangle \quad (45)$$

$$= e^{i\delta}\langle\mathbf{x}|\alpha\rangle \quad (46)$$

$$\implies \langle\mathbf{x}|\alpha\rangle = e^{-i\delta}\langle\mathbf{x}|\tilde{\alpha}\rangle \quad (47)$$

$$= e^{-i\delta}\psi_\alpha^*(\mathbf{x}) \quad (48)$$

$$= e^{-i\delta} \sum_{l,m} F_{l,m}^*(r) [Y_l^m(\theta, \phi)]^* \quad (49)$$

$$= e^{-i\delta} \sum_{l,m} F_{l,m}^*(r) (-1)^m Y_l^{-m}(\theta, \phi) \quad (50)$$

$$= e^{-i\delta} \sum_{l,m} F_{l,-m}^*(r) (-1)^{-m} Y_l^m(\theta, \phi) \quad (51)$$

Comparing the coefficients to  $Y_l^m$ :

$$F_{l,m} = (-1)^{-m} e^{-i\delta} F_{l,-m}^*(r) \quad (52)$$

4. The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2). \quad (53)$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics). Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?

For a spin 1 system the spin operators have the form

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (54)$$

$$S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (55)$$

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (56)$$

And so the Hamiltonian is:

$$H = AS_z^2 + B(S_x^2 - S_y^2) \quad (57)$$

$$= \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix} \quad (58)$$

The eigenvalues are  $0, A \pm B$ , with eigenvectors  $(0,1,0)$ ,  $(1,0,1)/\sqrt{2}$ ,  $(1,0,-1)/\sqrt{2}$ , respectively.

Or, in terms of  $|s, s_z\rangle$ :

$$\begin{cases} |1, 0\rangle & 0 \\ \frac{1}{\sqrt{2}} (|1, 1\rangle + |1, -1\rangle) & A + B \\ \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle) & A - B \end{cases} \quad (59)$$

Assume  $A$  and  $B$  are real (otherwise  $H$  is not Hermitian). Then:

$$\Theta H \Theta^{-1} = A \Theta S_z \Theta^{-1} \Theta S_z \Theta^{-1} + B (\Theta S_x \Theta^{-1} \Theta S_x \Theta^{-1} - \Theta S_y \Theta^{-1} \Theta S_y \Theta^{-1}) \quad (60)$$

$$= A (-S_z)^2 + B [(-S_x)^2 - (-S_y)^2] \quad (61)$$

$$= H \quad (62)$$

The Hamiltonian is invariant under time reversal.

We know that  $\Theta |lm\rangle = (-1)^m |l, -m\rangle$ , so the first and third eigenstates (eigenvalues 0 and  $A - B$ ) are invariant under time reversal, while the other (eigenvalue  $A + B$ ) is odd under time reversal

$$\begin{cases} \Theta |1, 0\rangle = |1, 0\rangle \\ \Theta \frac{1}{\sqrt{2}} (|1, 1\rangle + |1, -1\rangle) = -\frac{1}{\sqrt{2}} (|1, 1\rangle + |1, -1\rangle) \\ \Theta \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle) = \frac{1}{\sqrt{2}} (|1, 1\rangle - |1, -1\rangle) \end{cases} \quad (63)$$