Homework Set 3 — Symmetries 2

Due April 17

1. (a) Let $\psi(\mathbf{x}, t)$ be the wave function of a spinless particle corresponding to a plane wave in three dimensions. Show that $\psi^*(\mathbf{x}, -t)$ is the wave function for the plane wave with the momentum direction reversed.

   (b) Let $\chi(\hat{\mathbf{n}})$ be the two-component eigenspinor of $\sigma \cdot \hat{\mathbf{n}}$ with eigenvalue $+1$ (see Sakurai Ch. 3.2). Using the explicit form of $\chi(\hat{\mathbf{n}})$ (in terms of the polar and azimuthal angles β and γ corresponding to direction $\hat{\mathbf{n}}$) verify that $-\sigma_2 \chi^*(\hat{\mathbf{n}})$ is the two-component eigenspinor with the spin direction reversed.

2. (a) What is the time-reversed state corresponding to $\mathcal{D}(R)|j, m\rangle$?

   (b) Using the properties of time reversal and rotations, prove the matrix elements of the rotation operator satisfy

$$\mathcal{D}^{(j)}_{m'm}(R) \equiv \langle j, m'|\mathcal{D}(R)|j, m\rangle \tag{1}$$

$$= i^{-2(m'+m)} \mathcal{D}^{(j)}_{-m',-m}(R). \tag{2}$$

3. Suppose a spinless particle is bound to a fixed center by a 3D potential $V(\mathbf{x})$ that is so asymmetric, no energy level is degenerate. Use time-reversal invariance to prove

$$\langle \mathbf{L} \rangle = 0 \tag{3}$$

   for any energy eigenstate. (This is known as **quenching** of orbital angular momentum.) If the wave function of such a nondegenerate eigenstate is expanded as

$$\sum_l \sum_m F_{lm}(r) Y_l^m(\theta, \phi), \tag{4}$$

   what kind of phase restrictions do we obtain on $F_{lm}(r)$?

4. The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B(S_x^2 - S_y^2). \tag{5}$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics). Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?