

# Homework Set 2 — Symmetries 1

Due April 3

1. Parity is often treated as a mirror reflection. This is certainly true in 1 dimension, where  $x \rightarrow -x$  may be viewed as the effect of reflecting through a (point) mirror at the origin. In three dimensions when we use a plane mirror (say lying on the  $x - y$  plane), only one ( $z$ ) coordinate gets reversed, whereas the parity transformation reverses all three coordinates.  
Verify that reflection in a mirror in the  $x - y$  plane is the same as parity followed by a  $\pi$  ( $180^\circ$ ) rotation about the  $z$  axis. Since rotational invariance holds for weak interactions, noninvariance under mirror reflection implies noninvariance under parity.
2. Let  $\mathcal{T}_{\mathbf{d}}$  denote the translation operator (with displacement vector  $\mathbf{d}$ );  $\mathcal{D}(\hat{\mathbf{n}}, \phi)$  the rotation operator ( $\hat{\mathbf{n}}$  and  $\phi$  are the axis and angle of rotation, respectively); and  $\pi$  the parity operator. Which, if any, of the following pairs commute? Why?
  - (a)  $\mathcal{T}_{\mathbf{d}}$  and  $\mathcal{T}_{\mathbf{d}'}$  (with  $\mathbf{d}$  and  $\mathbf{d}'$  in different directions)
  - (b)  $\mathcal{D}(\hat{\mathbf{n}}, \phi)$  and  $\mathcal{D}(\hat{\mathbf{n}}', \phi)$  ( $\hat{\mathbf{n}}$  and  $\hat{\mathbf{n}}'$  in different directions)
  - (c)  $\mathcal{T}_{\mathbf{d}}$  and  $\pi$
  - (d)  $\mathcal{D}(\hat{\mathbf{n}}, \phi)$  and  $\pi$
3. A spin  $\frac{1}{2}$  particle is bound to a fixed center by a spherically symmetric potential.
  - (a) Write down the spin angular function  $\mathcal{Y}_{l=0}^{j=1/2, m=1/2}$  (see, e.g., Sakurai Chapter 3)
  - (b) Express  $(\boldsymbol{\sigma} \cdot \mathbf{x})\mathcal{Y}_{l=0}^{j=1/2, m=1/2}$  in terms of some other  $\mathcal{Y}_l^{j, m}$ .
  - (c) Show that your result in 3b is understandable in view of the transformation properties of the operator  $\mathbf{S} \cdot \mathbf{x}$  under rotations and under space inversion (parity)

4. Because of weak (neutral-current) interactions there is a parity-violating potential between the atomic electron and the nucleus as follows:

$$V = \lambda [\delta^{(3)}(\mathbf{x})\mathbf{S} \cdot \mathbf{p} + \mathbf{S} \cdot \mathbf{p}\delta^{(3)}(\mathbf{x})], \quad (1)$$

where  $\mathbf{S}$  and  $\mathbf{p}$  are the spin and momentum operators of the electron, and the nucleus is assumed to be situated at the origin. As a result, the ground state of an alkali atom, usually characterized by  $|n, l, j, m\rangle$  actually contains very tiny contributions from other eigenstates as follows:

$$|n, l, j, m\rangle \rightarrow |n, l, j, m\rangle + \sum_{n'l'j'm'} C_{n'l'j'm'} |n', l', j', m'\rangle. \quad (2)$$

On the basis of symmetry considerations *alone*, what can you say about  $(n', l', j', m')$ , which give rise to nonvanishing contributions? Suppose the radial wave functions and the energy levels are all known. Indicate how you may calculate  $C_{n'l'j'm'}$ . Do we get further restrictions on  $(n', l', j', m')$ ?