Solutions:

Homework Set 1 — Identical Particles

Due March 27

1. (a)

$$\begin{aligned}
\left[\Omega_1^{(1)} \otimes \mathbb{1}^{(2)}, \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)}\right] |\omega_1\rangle \otimes |\omega_2\rangle &= \Omega_1^{(1)} \otimes \mathbb{1}^{(2)} \mathbb{1}^{(1)} \otimes \Lambda_2(2) |\omega_1\rangle \otimes |\omega_2\rangle \\
&\quad - \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)} \Omega_1^{(1)} \otimes \mathbb{1}^{(2)} |\omega_1\rangle \otimes |\omega_2\rangle \qquad (1) \\
&= \Omega_1^{(1)} \otimes \mathbb{1}^{(2)} |\mathbb{1}^{(1)} \omega_1\rangle \otimes |\Lambda_2^{(2)} \omega_2\rangle \\
&\quad - \mathbb{1}^{(1)} \otimes \Lambda_2^{(2)} |\Omega_1^{(1)} \omega_2\rangle \otimes |\mathbb{1}^{(2)} \omega_2\rangle \qquad (2) \\
&= |\Omega_1^{(1)} \omega_1\rangle \otimes |\mathbb{1}^{(1)} \Lambda_2^{(2)} \omega_2\rangle - |\mathbb{1}^{(1)} \Omega_1^{(1)} \omega_1\rangle \otimes |\Lambda_2^{(2)} \omega_2\rangle \qquad (3) \\
&= |\Omega_1^{(1)} \omega_1\rangle \otimes |\Lambda_2^{(2)} \omega_2\rangle - |\Omega_1^{(1)} \omega_1\rangle \otimes |\Lambda_2^{(2)} \omega_2\rangle \qquad (4) \\
&= 0 \qquad (5)
\end{aligned}$$

(b)

$$\begin{aligned}
\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right) \left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right) |\omega_1\rangle \otimes |\omega_2\rangle &= \left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right) |\theta_1^{(1)} \omega\rangle \otimes |\Lambda_2^{(2)} \omega_2\rangle \qquad (6) \\
&= |\Omega_1^{(1)} \theta_1^{(1)} \omega_1\rangle \otimes |\Gamma_2^{(2)} \Lambda_2^{(2)} \omega_2\rangle \qquad (7) \\
&= \left(\Omega_1^{(1)} \theta_1^{(1)}\right) \otimes \left(\Gamma_2^{(2)} \Lambda_2^{(2)}\right) |\omega_1\rangle \otimes |\omega_2\rangle \qquad (8) \\
&= \left[(\Omega\theta)^{(1)} \otimes (\Gamma\Lambda)^{(2)}\right] |\omega_1\rangle \otimes |\omega_2\rangle \qquad (9) \\
\left(\Omega_1^{(1)} \otimes \Gamma_2^{(2)}\right) \left(\theta_1^{(1)} \otimes \Lambda_2^{(2)}\right) &= (\Omega\theta)^{(1)} \otimes (\Gamma\Lambda)^{(2)} \qquad (10)
\end{aligned}$$

(c)

$$\left[\Gamma_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right]|\omega_1\rangle\otimes|\omega_2\rangle = \left[\Gamma_1^{(1)}\otimes\mathbb{1}^{(2)}, \Lambda_1^{(1)}\otimes\mathbb{1}^{(2)}\right]|\omega_1\rangle\otimes|\omega_2\rangle \tag{11}$$

$$= |\left[\Gamma_1^{(1)}, \Lambda_1^{(1)}\right]\omega_1\rangle\otimes|\mathbb{1}^{(2)}\omega_2\rangle \tag{12}$$

$$= |\Gamma_1^{(1)}\omega_1\rangle\otimes|\mathbb{1}^{(2)}\omega_2\rangle \tag{13}$$

$$= \Gamma_1^{(1)}\otimes\mathbb{1}^{(2)}|\omega_1\rangle\otimes|\omega_2\rangle \tag{14}$$

$$\left[\Gamma_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}\right] = \Gamma_1^{(1)}\otimes\mathbb{1}^{(2)} \tag{15}$$

(d)

$$\left(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)}\right)^2 = \left(\Omega_1^{(1)}\otimes\mathbb{1}^{(2)} + \mathbb{1}^{(2)}\otimes\Omega_2^{(1)}\right)^2|\omega_1\rangle\otimes\omega_2\rangle \tag{16}$$

$$\begin{aligned} &= \left(\Omega_1^{(1)}\otimes\mathbb{1}^{(2)}\right)^2|\omega_1\rangle\otimes\omega_2\rangle \\ &\quad + \Omega_1^{(1)}\otimes\mathbb{1}^{(2)}\mathbb{1}^{(1)}\otimes\Omega_2^{(2)}|\omega_1\rangle\otimes\omega_2\rangle \\ &\quad + \mathbb{1}^{(1)}\otimes\Omega_2^{(2)}\Omega_1^{(1)}\otimes\mathbb{1}^{(2)}|\omega_1\rangle\otimes\omega_2\rangle \\ &\quad + \left(\mathbb{1}^{(1)}\otimes\Omega_2^{(2)}\right)^2|\omega_1\rangle\otimes\omega_2\rangle \end{aligned} \tag{17}$$

$$\begin{aligned} &= |(\Omega_1^2)^{(1)}\omega_1\rangle\otimes|\mathbb{1}^{(2)}\omega_2\rangle + |\Omega_1^{(1)}\omega_1\rangle\otimes|\Omega_2^{(2)}\omega_2\rangle \\ &\quad + |\Omega_1^{(1)}\omega_1\rangle\otimes|\Omega_2^{(2)}\omega_2\rangle + |\mathbb{1}^{(1)}\omega_1\rangle\otimes|(\Omega_2^2)^{(2)}\omega_2\rangle \\ &= \left(\left(\Omega_1^2\right)^{(1)}\otimes\mathbb{1}^{(2)} + 2\Omega_1^{(1)}\otimes\Omega_2^{(2)} + \mathbb{1}^{(1)}\otimes\left(\Omega_1^2\right)^{(1)}\right)|\omega_1\rangle\otimes\omega_2\rangle \end{aligned} \tag{18}$$

$$\left(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)}\right)^2 = \left(\Omega_1^2\right)^{(1)}\otimes\mathbb{1}^{(2)} + \mathbb{1}^{(1)}\otimes\left(\Omega_1^2\right)^{(1)} + 2\Omega_1^{(1)}\otimes\Omega_2^{(2)} \tag{19}$$

2. ...

3. For a single particle in a 1D SHO potential the energy spectrum is

$$E_n^{(1)} = \hbar\omega\left(n + \frac{1}{2}\right); \qquad n = 0, 1, 2, \ldots \tag{20}$$

Each energy state can accommodate a maximum 2 spin $\frac{1}{2}$ particles (one for each possible spin state, $2s+1$).
E.g., for 3 particles, two can be in the lowest energy $E_0^{(1)}$ and the third must be in the

next higher state $E_1^{(1)}$, and the ground state is

$$E_0^{(3)} = 2\left(\frac{1}{2}\hbar\omega\right) + 1\left(\frac{3}{2}\hbar\omega\right) = \frac{5}{2}\hbar\omega \tag{21}$$

For an arbitrary number N particles, when N is even, each occupied state contains two particles and the total ground state energy is

$$E_0^{(N)} = 2\sum_{n=0}^{\frac{N}{2}-1} \hbar\omega(n + \frac{1}{2}) \tag{22}$$

$$= \frac{\hbar\omega}{4}N^2. \tag{23}$$

When N is odd, the highest energy single-particle state is singly occupied:

$$E_0^{(N)} = 2\sum_{n=0}^{\frac{N-3}{2}} \hbar\omega(n + \frac{1}{2}) + \hbar\omega(\frac{N}{2} - \frac{1}{2} + \frac{1}{2}) \tag{24}$$

$$= \frac{\hbar\omega}{4}(N^2 + 1). \tag{25}$$

or, summarizing

$$E_0^{(N)} = \begin{cases} \frac{N^2}{4}\hbar\omega, & N = \text{even} \\ \frac{N^2+1}{4}\hbar\omega, & N = \text{odd} \end{cases} \tag{26}$$

The Fermi energy is the highestoccupied single-particle state in the N-particle ground state. The highest occupied state is $E_{(N-1)/2}^{(1)}$ for odd N and $E_{(N-2)/2}^{(1)}$ for even N. So the Fermi energy is

$$E_F^{(N)} = \begin{cases} \frac{N-1}{2}\hbar\omega & N = \text{even} \\ \frac{N}{2}\hbar\omega & N = \text{odd} \end{cases} \tag{27}$$

In the limit $N \to \infty$, we have

$$\lim_{N\to\infty} E_0^{(N)} = \frac{N^2}{4}\hbar\omega, \tag{28}$$

$$\lim_{N\to\infty} E_F^{(N)} = \frac{N}{2}\hbar\omega. \tag{29}$$

4. Spin 1 particles are bosons and must have symmetric wavefunction. They are in the same orbital state, and so they must also be in a symmetric spin state.

   The $j = 0$ state and the (5) $j = 2$ states are symmetric with respect to interchange of particles, while the (3) $j = 1$ states are antisymmetric. Thus, only states with $j = 0$ or $j = 2$ are allowed.

One can see this explicitly by constructing the full states. For example, the stretch state $|j = 2, m = 0\rangle$ must be symmetric, $|+\rangle|+\rangle$, and the other $j = 2$ states can be obtained from the lowering operator $S_- = S_{1-} + S_{2-}$:

$$
\begin{aligned}
|j = 2, m = 2\rangle &= |+\rangle|+\rangle && (30) \\
|j = 2, m = 1\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle|0\rangle + |0\rangle|+\rangle\right) && (31) \\
|j = 2, m = 0\rangle &= \frac{1}{\sqrt{6}}\left(2|0\rangle|0\rangle + |-\rangle|+\rangle - |+\rangle|-\rangle\right) && (32) \\
|j = 2, m = -1\rangle &= \frac{1}{\sqrt{2}}\left(|-\rangle|0\rangle + |0\rangle|-\rangle\right) && (33) \\
|j = 2, m = -2\rangle &= |-\rangle|-\rangle. && (34) \\
& && (35)
\end{aligned}
$$

These are all symmetric.

The $|j = 1, m = 1\rangle$ state must be orthogonal to the $|j = 2, m = 1\rangle$ state, and can be similarly lowered to obtain the other states:

$$
\begin{aligned}
|j = 1, m = 1\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle|0\rangle - |0\rangle|+\rangle\right) && (36) \\
|j = 1, m = 0\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right) && (37) \\
|j = 1, m = -1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|-\rangle - |-\rangle|0\rangle\right) && (38)
\end{aligned}
$$

These states are all antisymmetric.

The $|j = 0, m = 0\rangle$ state must be orthogonal to the other $m = 0$ states:

$$
|j = 0, m = 0\rangle = \frac{1}{\sqrt{3}}\left(|+\rangle|-\rangle - |0\rangle|0\rangle + |-\rangle|+\rangle\right) \qquad (39)
$$

This state is symmetric with respect to particle interchange.

Or, we can see

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{1,1} \oplus \ydiagram{2} \tag{40}$$

$$3 \times 3 = 3 + 6 \tag{41}$$

There are 3 antisymmetric states and 6 symmetric states. We know there must be 5 $j = 2$ states, 3 $j = 1$ states, and 1 singlet state. So the antisymmetric states must represent the $j = 1$ triplet, and are not allowed.

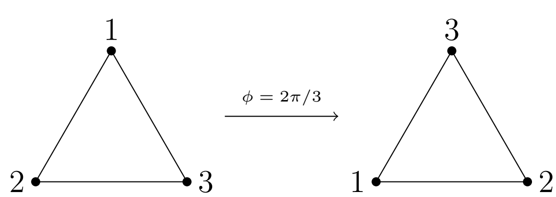
5. For two spinless bosons, the orbital wavefunction is necessarily symmetric under interchange of the particles

   If we neglect spin interactions, then the energy levels would be the same for $\frac{1}{2}$ or spin 0 "electrons". However, excited states with antisymmetric wavefunctions are not allowed, and would disappear from the energy spectrum. Specifically, all the spin-triplet states (orthohelium) would disappear.

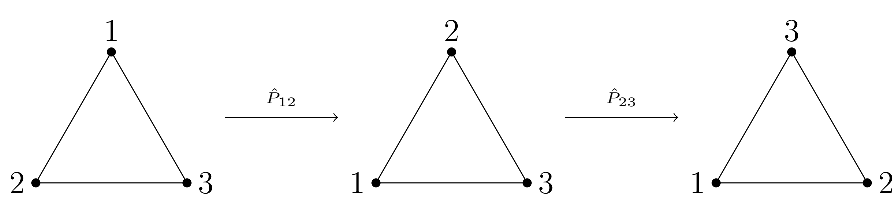
6. The rotation operator around the z-axis by angle ϕ can be written

$$D(\phi) = e^{-i\hat{J}_z\phi/\hbar}. \tag{42}$$

   Rotations by multiplies of $\phi = 2\pi/3$ are equivalent to a series of particle exchanges, and must therefore leave the wavefunction invariant (since the particles are bosons). E.g., a rotation by $2\pi/3$



   is equivalent to the consecutive interchange $\hat{P}_{23}\hat{P}_{12}$



So it must be an eigenstate of $D(\phi = 2\pi/3)$ with eigenvalue 1. It must then be an eigenstate of $\hat{J}_z$:

$$
\begin{aligned}
D(\phi = 2\pi/3)|\alpha\rangle &= e^{-i\hat{J}_z\phi/\hbar}|\alpha\rangle && (43)\\
&= e^{-im*2\pi/3}|\alpha\rangle && (44)\\
&= |\alpha\rangle, && (45)
\end{aligned}
$$

and the state must have $m = 0, \pm 3, \pm 6, \ldots$, or $\frac{m}{3} \in \mathbb{Z}$

7. (a) The space part of the state vector is symmetric $\implies$ the spin part must be symmetric under interchange of any pair

    i. The symmetric state is trivially

$$|+\rangle|+\rangle|+\rangle \tag{46}$$

    This has total spin $s = 3$ (and $m = 3$)

    ii. The only completely symmetric combination is

$$\frac{1}{\sqrt{3}}\left(|+\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|+\rangle + |0\rangle|+\rangle|+\rangle\right) \tag{47}$$

    This is the $s = 3$, $m = 2$ state.

    iii. The completely symmetric combination here is

$$\frac{1}{\sqrt{6}}\left(|+\rangle|0\rangle|-\rangle + |+\rangle|-\rangle|0\rangle + |0\rangle|+\rangle|-\rangle + |-\rangle|+\rangle|0\rangle + |0\rangle|-\rangle|+\rangle + |-\rangle|0\rangle|+\rangle\right) \tag{48}$$

    This is not an eigenstate of S^2, and so does not have a definite total spin. It is a linear combination of $m = 0$ states with different s. One can check this by acting on the state with the S^2 operator, and note that an extra term appears:

$$S^2|\Psi\rangle = 12|\Psi\rangle + \frac{24}{\sqrt{6}}|0\rangle|0\rangle|0\rangle \tag{49}$$

   (b) For 7(a)iii, there is one completely antisymmetric state:

$$\frac{1}{\sqrt{6}}\left(|+-0\rangle - |+-0\rangle + |0-+\rangle - |0+-\rangle + |-+0\rangle - |-0+\rangle\right) \tag{50}$$

   This is the spin singlet state $s = 0$.

8. For each spin $\frac{3}{2}$ electron there are four possible spin orientations. Therefore it is possible to put four electrons in each spatial state.

   The lowest energy configuration is then:

$$(1s)^4(2s)^4(2p)^2 \tag{51}$$

   This state is degenerate because each of the two electrons in the p state can have $m_l = 1, 0, -1$ and $m_s = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

   A single particle has 12 possible states, and so two indistinguishable can occupy these states a total of

$$\binom{12}{2} = \frac{12!}{2!(12-2)!} = 66 \tag{52}$$

   different ways, compared to the single ground state configuration of the standard case.

   All of these are allowed, since we can make an antisymmetric state for any two distinct choices for $(m_l,\, m_s)$:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_{m_l,m_s}\rangle|\Psi_{m_l',m_s'}\rangle - |\Psi_{m_l',m_s'}\rangle|\Psi_{m_l,m_s}\rangle\right) \tag{53}$$

   This degeneracy is broken when you take into account spin-orbit and exchange splitting. The piece of the Hamiltonian responsible for the spin-orbit interaction is proportional to

$$\hat{L}\cdot\hat{S} = \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2) \tag{54}$$

   $\therefore$ the lowest energy state is when S and L are maximized while J is minimized.

   The total spin of two spin $\frac{3}{2}$ particles can be S = 3,2,1, or 0, while the total orbital angular momentum can be L = 2, 1, or 0. $(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)\Psi$ is minimized for S=3, L=2, J=1. However, this is not allowed, because the wavefunction must be antisymmetric.

   With this restriction the minimum is for S=3, L=1, J=2. This also agrees with the effects of exchange splitting – the lowest energy states have a symmetric spin state and antisymmetric spatial state. The new ground state is written $^7P_2$