Solutions:
Homework Set 9

Due October 29, 2020

1. Consider the wave function of a particle with spin (not necessarily $s = 1/2$) moving in 3 dimensions.

$$\psi(\mathbf{x}, m) = \langle \mathbf{x}, m|\psi\rangle = \psi_m(\mathbf{x}) \tag{1}$$

   (a) If $\psi_m(\mathbf{r})$ is the wave function of a particle in state $|\psi\rangle$, then what is the wave function of the particle in the rotated state $U(R)|\psi\rangle$? The rotation operator $U(R)$ is the product of a spatial rotation times a spin rotation, both parameterized by the same R.

   (b) Consider the specific case of a spin-1/2 electron moving in an electromagnetic field, plus a central force potential $V(r)$. The full Hamiltonian is

$$H = \frac{1}{2m}\left[\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{x}, \mathbf{t})\right]^2 + q\Phi(\mathbf{x}, t) - \mu \cdot \mathbf{B}(\mathbf{x}, t) + V(r). \tag{2}$$

   For an electron we have $q = -e$ and $g = g_e \simeq 2$, so that $\mu = -g_e\mu_B\mathbf{S}/\hbar$.

   Let the magnetic field be uniform, $\mathbf{B} = B\hat{b}$, and choose a gauge such that $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{x}$ and $\Phi = 0$. Let $\omega_0 = eB/mc$.

   Consider the time-dependent Schrödinger equation for the electron,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{3}$$

   Define a new state $|\phi(t)\rangle$ by

$$|\psi(t)\rangle = U(\hat{b}, \omega t)|\phi(t)\rangle, \tag{4}$$

   where $U(\hat{b}, \omega t)$ is a rotation operator that rotates the whole system (orbital and spin degrees of freedom). This means that $|\phi(t)\rangle$ is the state in a frame rotating with angular velocity ω about the axis $\hat{b}$.

Find a frequency ω that eliminates the effect of the magnetic field on the orbital motion of the particle, apart from the centrifugal potential which is proportional to $(\mathbf{b} \times \mathbf{x})^2$. Find a frequency ω that eliminates the effect of the magnetic field on the spin.

Express your answers as some multiple of ω_0. Can you eliminate the effects of the magnetic field entirely, apart from the centrifugal potential?

(a) $$|\psi\rangle = \sum_m \int d^3\vec{r} \; |\vec{r} m\rangle \, \psi_m(\vec{r})$$

$$U(R)|\psi\rangle = \sum_m \int d^3\vec{r} \; U(R) \, |\vec{r}\rangle |m\rangle \, \psi_m(\vec{r})$$

$$= \sum_m \int d^3\vec{r} \; |R\vec{r}\rangle \sum_{m'} |m'\rangle D_{m'm}(R) \, \psi_m(\vec{r})$$

$R\vec{r} = \vec{r}\,'$

$$= \sum_{mm'} \int d^3\vec{r}\,' \; |\vec{r}\,'\rangle |m'\rangle \, D_{m'm}(R) \, \psi_m(R^{-1}\vec{r}\,')$$

$$= \sum_{mm'} \int d^3\vec{r} \; |\vec{r} m\rangle \, D_{mm'}(R) \, \psi_{m'}(R^{-1}\vec{r})$$ drop prime on $\vec{r}$, swap m m'

$$= \sum_m \int d^3\vec{r} \; |\vec{r} m\rangle \, \psi'_m(\vec{r})$$ ψ' = rotated ψ

$$\Rightarrow \quad \boxed{\psi'_m(\vec{r}) = \sum_{m'} D_{mm'}(R) \, \psi_{m'}(R^{-1}\vec{r})}$$

(b) $$i\hbar \frac{\partial}{\partial t} |\psi\rangle = i\hbar \frac{\partial}{\partial t} \left[U(\hat{b}, \omega t) \, |\phi\rangle \right]$$

$$= U \left[\omega \hat{b} \cdot \vec{J} + i\hbar \frac{\partial}{\partial t} \right] |\phi\rangle = H \, U |\phi\rangle$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} |\phi(t)\rangle = H' \, |\phi(t)\rangle$$ where

$$H' = U^{\dagger} H U - \omega \hat{b} \cdot \vec{J}$$

but

$$H = \frac{1}{2m}\left(\vec{p} + \frac{eB}{2c}\,\hat{b}\times\vec{r}\right)^2 + V(r) + g\,\frac{eB}{2mc}\,\hat{b}\cdot\vec{S}$$

$$= \frac{p^2}{2m} + \frac{\omega_0}{2}\hat{b}\cdot\vec{L} + \frac{m\omega_0^2}{8}(\hat{b}\times\vec{r})^2 + V(r) + \omega_0\,\hat{b}\cdot\vec{S}$$

using $g \approx 2$. Now,

$$U^+\vec{p}\,U = R\vec{p}$$
$$U^+\vec{r}\,U = R\vec{r}$$
$$U^+\vec{L}\,U = R\vec{L}$$
$$U^+\vec{S}\,U = R\vec{S}$$

and $R\hat{b} = \hat{b}$. So, $\quad U^+\frac{p^2}{2m}U^+ = \frac{p^2}{2m}$

$$U^+(\hat{b}\cdot\vec{L})U^+ = \hat{b}\cdot(R\vec{L}) = (R^{-1}\hat{b})\cdot\vec{L} = \hat{b}\cdot\vec{L}$$

$$U^+(\hat{b}\times\vec{r})U^+ = \hat{b}\times(R\vec{r}) = (R\hat{b})\times(R\vec{r}) = R\,(\hat{b}\times\vec{r})$$

$$U^+(\hat{b}\times\vec{r})^2\,U^+ = (\hat{b}\times\vec{r})^2$$

similarly, $\quad U^+V(r)\,U^+ = V(r)$

$$U^+\,\hat{b}\cdot\vec{S}\,U^+ = \hat{b}\cdot\vec{S}$$

and $U^+HU = H$, so $\quad H' = H - \omega\,\hat{b}\cdot\vec{J}, \qquad \vec{J} = \vec{L}+\vec{S}$

$$= \frac{p^2}{2m} + \frac{m\omega_0^2}{8}(\hat{b}\times\vec{r})^2 + V(r) + \left(\frac{\omega_0}{2} - \omega\right)\hat{b}\cdot\vec{L} + (\omega_0 - \omega)\,\hat{b}\cdot\vec{S}$$

Set $\omega = \frac{\omega_0}{2}$ to cancel $\hat{b}\cdot\vec{L}$ term, (you can't cancel them both.)

$\omega = \omega_0$ to cancel $\hat{b}\cdot\vec{S}$ term.

2. Add angular momenta $j_1 = 1$ and $j_2 = 1$ (e.g., two spin-1 particles, or a spin-1 particle in a $l = 1$ orbital state).

   What are the possible values for total angular momentum j?

   Express all possible $|j, m\rangle$ eigenkets in terms of tensor product eigenkets $|j_1 m_1; j_2 m_2\rangle$. Write your answer as

$$|j = 1, m = 1\rangle = \#|+, 0\rangle + \#|0, +\rangle, \ldots, \tag{5}$$

   where $+$, 0, and $-$ stand for $m_{1,2} = 1, 0, -1$ respectively.

The possible values are $j = 2, 1, 0$, for a total number of states $3 \times 3 = 5 + 3 + 1$, or

$$1 \otimes 1 = 2 \oplus 1 \oplus 0 \tag{6}$$

Lowering from the stretch state of $j = 2$, $|2, 0\rangle = |++\rangle$, we obtain

$$|j = 2, m = 2\rangle = |++\rangle \tag{7}$$

$$|2, 1\rangle = \frac{1}{2} J_- |2, 2\rangle = \frac{1}{\sqrt{2}} \big[|+0\rangle + |0+\rangle\big] \tag{8}$$

$$|2, 0\rangle = \frac{1}{\sqrt{6}} \big[|+-\rangle + |-+\rangle + 2|00\rangle\big] \tag{9}$$

$$|2, -1\rangle = \frac{1}{\sqrt{2}} \big[|-0\rangle + |0-\rangle\big] \tag{10}$$

$$|2, -2\rangle = |--\rangle \tag{11}$$

The state $|j = 1, m = 1\rangle$ must have $m_1 + m_2 = 1$, but be orthogonal to $|2, 2\rangle$. The phase convention is that the $m_1 = 1$ state has positive coefficient. Once we have that state, we can again use J_- to find the remaining states. So,

$$|11\rangle = \frac{1}{\sqrt{2}} \big[|+0\rangle - |0+\rangle\big] \tag{12}$$

$$|10\rangle = \frac{1}{\sqrt{2}} \big[|+-\rangle - |-+\rangle\big] \tag{13}$$

$$|1, -1\rangle = \frac{1}{\sqrt{2}} \big[|0-\rangle - |-0\rangle\big] \tag{14}$$

Finally, the $|j = 0, m = 0\rangle$ state must be orthogonal to the other two $m = 0$ states:

$$|00\rangle = \frac{1}{\sqrt{3}}\big[| + -\rangle + | - +\rangle - |00\rangle\big] \tag{15}$$