Solutions:

Homework Set 8

Due October 22, 2020

1. Consider central force motion in two dimensions. The wave function is $\psi(x, y) = \psi(\rho, \phi)$, where (ρ, ϕ) are plane polar coordinates with $x = \rho\cos\phi, \quad y = \rho\sin\phi$

   (a) Find the most general function $\psi(\rho, \phi)$ that is an eigenfunction of L_z (i.e., the generator of rotations in the $x - y$ plane). Express it in terms of a radial wave function $R(\rho)$, as done in class for the three-dimensional case. What is the spectrum of L_z?

$$L_z = xp_y - yp_x \tag{1}$$

$$= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{2}$$

$$= -i\hbar\frac{\partial}{\partial\phi} \tag{3}$$

when you transform to (ρ, ϕ) coordinates.

Let $\psi(x, y) = \psi(\rho, \phi)$. Then

$$L_z\psi == i\hbar\frac{\partial\psi}{\partial\phi} \tag{4}$$

$$= m\hbar\psi \tag{5}$$

where $m\hbar$ is the eigenvalue.

This has the general solution

$$\psi(\rho, \phi) = R(\rho)e^{im\phi}, \tag{6}$$

with $R(\rho)$ an arbitrary (complex) function of ρ and $m \in \mathbb{Z}$ so that the ψ is single valued $(\psi(\rho, \phi) = \psi(\rho, \psi + 2\pi))$.

So the spectrum of L_z is $\{m\hbar, m \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers.

(b) Consider a central force Hamiltonian in two dimensions,

$$H = \frac{\mathbf{p}^2}{2m} + V(\rho), \tag{7}$$

where $\mathbf{p} = (p_x, p_y)$. Show that this Hamiltonian commutes with L_z. Therefore simultaneous eigenfunctions of H and L_z exist.

By expressing the Laplacian in polar coordinates and using the result of part 1a, find a radial wave equation for $R(\rho)$ that will determine energy eigenfunctions and eigenvalues.

In polar coordinates

$$\begin{aligned}
L_z H\psi &= -i\hbar\frac{\partial}{\partial\phi}\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\rho)\right)\psi && (8)\\
&= -i\hbar\frac{\partial}{\partial\phi}\left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{\hbar^2}{2m}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + V(\rho)\right)\psi && (9)\\
&= \left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{\hbar^2}{2m}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + V(\rho)\right)\left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi && (10)\\
&= HL_z\psi, && (11)
\end{aligned}$$

since $\partial/\partial\phi$ commutes with $\partial/\partial\rho$.

Using 1a, we have

$$\begin{aligned}
H\psi &= E\psi && (12)\\
&= \left(-\frac{\hbar^2}{2m\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} - \frac{\hbar^2}{2m}\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + V(\rho)\right)R(\rho)e^{in\phi} && (13)\\
&= \left(-\frac{\hbar^2}{2m\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{\hbar^2}{2m}\frac{n^2}{\rho^2} + V(\rho)\right)R(\rho)e^{in\phi} && (14)\\
&= ER(\rho)e^{n\phi} && (15)\\
\implies 0 &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} - \frac{n^2}{\rho^2}R - \frac{2m}{\hbar^2}[V(\rho) - E]R && (16)
\end{aligned}$$

(c) Define a modified radial wave function by

$$f(\rho) = \rho^a R(\rho), \tag{17}$$

where a is a power to be determined. Determine a by requiring that the modified radial wave equation should look like the one-dimensional Schrödinger equation, apart from the range of the variable ρ and the presence of the centrifugal potential.

$$R(\rho) = \rho^{-a} f(\rho) \tag{18}$$

$$\frac{dR}{d\rho} = R' = -a\rho^{-a-1} f + \rho^{-a} f' \tag{19}$$

$$= \rho^{-a}\left(-\frac{a}{\rho} f + f'\right) \tag{20}$$

$$\rho\frac{dR}{d\rho} = \rho^{-a}\left(-af + \rho f'\right) \tag{21}$$

$$\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = -a\rho^{-a-1}\left(-af + \rho f'\right) + \rho^{-a}\left(-af' + f' + \rho f''\right) \tag{22}$$

$$= \rho^{-a}\left(\frac{a^2 f}{\rho} - 2af' + f' + \rho f''\right) \tag{23}$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \rho^{-a}\left(\frac{a^2 f}{\rho^2} + \frac{(-2a+1)f'}{\rho} + f''\right) \tag{24}$$

So if we want something that looks like the Schrodinger equation in 1D, we can't have any first derivatives (f' terms). So we must have $a = 1/2$. Then

$$f(\rho) = \sqrt{\rho} R(\rho) \tag{25}$$

and the normalization condition for $f(\rho)$ is also that of a 1D problem, other than the range of the integral

$$\int_0^\infty \rho d\rho |R(\rho)|^2 = \int_0^\infty d\rho |f(\rho)|^2 \tag{26}$$

The full equation is then

$$-\frac{\hbar^2}{2m}\rho^{-1/2}\left(\frac{\frac{1}{4}f}{\rho^2} + f''\right) + \left[\frac{\hbar^2 n^2}{2m\rho^2} + V(\rho)\right]\rho^{-1/2} f = E\rho^{-1/2} f \tag{27}$$

or, canceling the factor of $\rho^{-1/2}$ and simplifying,

$$-\frac{\hbar^2}{2m}\frac{d^2 f}{d\rho^2} + \left[\frac{\hbar^2(n^2 - 1/4)}{2m\rho^2} + V(\rho)\right] f = Ef \tag{28}$$

(d) Consider the case of the free particle. Express the radial eigenfunctions $R(\rho)$ in terms of ordinary Bessel functions, $J_\nu(x)$, and in terms of the energy and quantum number of L_z. You will have to look up the differential equation for ordinary Bessel functions.

For a free particle $V = 0$:

$$0 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} - \frac{n^2}{\rho^2}R + \frac{2m}{\hbar^2}ER \tag{29}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \left(k^2 - \frac{n^2}{\rho^2}\right)R \tag{30}$$

with $k^2 = \frac{2m}{\hbar^2}E$. Let $x = k\rho$. Then

$$0 = \frac{k^2}{x}\frac{\partial}{\partial x}x\frac{\partial R}{\partial x} + \left(k^2 - k^2\frac{n^2}{x^2}\right)R \tag{31}$$

$$0 = \frac{1}{x}\frac{\partial}{\partial x}x\frac{\partial R}{\partial x} + \left(1 - \frac{n^2}{x^2}\right)R \tag{32}$$

or

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - n^2)R = 0 \tag{33}$$

The solutions to this equation are the Bessel functions $J_n(x)$ and Neumann functions, $N_n(x)$ (sometimes written $Y_n(x)$), but the Neumann functions diverge at the origin:

$$\lim_{x\to 0} N_n(x) = -\infty, \tag{34}$$

so we know that our solution is proportional to J_n:

$$R(x) = R(k\rho) = J_n(k\rho) \tag{35}$$

where n is the quantum number associated with L_z.

(e) For any potential that is not too badly behaved near the origin, find the dependence of the radial eigenfunction $R(\rho)$ near $\rho = 0$. Verify that the free particle solutions of part 1d satisfy this condition.

$$R(x) = R(k\rho) = J_n(k\rho) \tag{36}$$

Near $\rho = 0$, let $R(\rho)) = a\rho^\nu$. Then

$$\frac{dR}{d\rho} = \nu a \rho^{\nu-1} \tag{37}$$
$$\rho \frac{dR}{d\rho} = \nu a \rho^{\nu} \tag{38}$$
$$\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = \nu^2 a \rho^{\nu-1} \tag{39}$$
$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = \nu^2 a \rho^{\nu-2} \tag{40}$$

So then

$$0 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho \frac{\partial R}{\partial \rho} - \frac{n^2}{\rho^2}R - \frac{2m}{\hbar^2}[V(\rho) - E]R \tag{41}$$
$$= \nu^2 a \rho^{\nu-2} - n^2 a \rho^{\nu-2} - \frac{2m}{\hbar^2}[V(\rho) - E]a\rho^\nu \tag{42}$$

Assuming V is well behaved, at small ρ this equation is dominated by the first two terms:

$$0 = (\nu^2 - n^2)a\rho^{\nu-2} \tag{43}$$
$$\implies \nu = \pm n \tag{44}$$

However, $\nu = -n$ is not allowed because the wavefunction would not be normalizable. Near $\rho = 0$, the normalization integral would be

$$\int \rho d\rho |R(\rho)|^2 = \int \rho d\rho \frac{a}{\rho^{2n}}, \tag{45}$$

which diverges for all $n > 0$.

The Bessel functions $J_n(k\rho) \sim \rho^n$ near $\rho = 0$, as expected.