Homework Set 8

Due October 22, 2020

1. Consider central force motion in two dimensions. The wave function is $\psi(x, y) = \psi(\rho, \phi)$, where (ρ, ϕ) are plane polar coordinates with $x = \rho \cos\phi, \quad y = \rho \sin\phi$

   (a) Find the most general function $\psi(\rho, \phi)$ that is an eigenfunction of L_z (i.e., the generator of rotations in the $x - y$ plane). Express it in terms of a radial wave function $R(\rho)$, as done in class for the three-dimensional case. What is the spectrum of L_z?

   (b) Consider a central force Hamiltonian in two dimensions,

   $$H = \frac{\mathbf{p}^2}{2m} + V(\rho), \tag{1}$$

   where $\mathbf{p} = (p_x, p_y)$. Show that this Hamiltonian commutes with L_z. Therefore simultaneous eigenfunctions of H and L_z exist.

   By expressing the Laplacian in polar coordinates and using the result of part 1a, find a radial wave equation for $R(\rho)$ that will determine energy eigenfunctions and eigenvalues.

   (c) Define a modified radial wave function by

   $$f(\rho) = \rho^a R(\rho), \tag{2}$$

   where a is a power to be determined. Determine a by requiring that the modified radial wave equation should look like the one-dimensional Schrödinger equation, apart from the range of the variable ρ and the presence of the centrifugal potential.

   (d) Consider the case of the free particle. Express the radial eigenfunctions $R(\rho)$ in terms of ordinary Bessel functions, $J_\nu(x)$, and in terms of the energy and quantum number of L_z. You will have to look up the differential equation for ordinary Bessel functions.

   (e) For any potential that is not too badly behaved near the origin, find the dependence of the radial eigenfunction $R(\rho)$ near $\rho = 0$. Verify that the free particle solutions of part 1d satisfy this condition.