

Solutions:

Homework Set 7

Due October 15, 2020

1. Show that the 3×3 matrices $G_i (i = 1, 2, 3)$ whose elements are given by

$$(G_i)_{jk} = -i\hbar\epsilon_{ijk}, \quad (1)$$

where j and k are the row and column indices, satisfy the angular-momentum commutation relations.

What is the physical (or geometric) significance of the transformation matrix that connects G_i to the more usual 3×3 representations of the angular-momentum operator J_i with J_3 taken to be diagonal?

Relate your result to

$$\mathbf{V} \rightarrow \mathbf{V} + \hat{\mathbf{n}}\delta\phi \times \mathbf{V} \quad (2)$$

under infinitesimal rotations. (*Note:* This problem may be helpful in understanding the photon spin.)

It is useful to note the following identity of the Levi-Civita symbol

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \quad (3)$$

$$= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \quad (4)$$

of which a special case is

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \quad (5)$$

So the commutator is

$$[G_i, G_j]_{ln} = (G_i G_j - G_j G_i)_{lm} = (G_i)_{lm} (G_j)_{mn} - (G_j)_{lm} (G_i)_{mn} \quad (6)$$

$$= -\hbar^2 [\epsilon_{ilm} \epsilon_{jmn} - \epsilon_{jlm} \epsilon_{imn}] \quad (7)$$

$$= -\hbar^2 [(\delta_{in} \delta_{lj} - \delta_{ij} \delta_{ln}) - (\delta_{jn} \delta_{li} - \delta_{ji} \delta_{ln})] \quad (8)$$

$$= \hbar^2 (\delta_{jn} \delta_{li} - \delta_{in} \delta_{lj}) \quad (9)$$

$$= \hbar^2 \epsilon_{kij} \epsilon_{kln} \quad (10)$$

$$= i\hbar \epsilon_{ijk} (G_k)_{ln} \quad (11)$$

Which is the usual commutation relation for angular momentum

$$[G_i, G_j] = i\hbar \epsilon_{ijk} G_k. \quad (12)$$

The usual representation of J_i has J_3 diagonal. We can obtain this representation by finding the unitary transformation that diagonalizes G_3 :

$$G_3 = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (13)$$

The eigenvalues and eigenvectors are

$$\lambda = +\hbar, \quad \vec{r}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad (14)$$

$$\lambda = 0, \quad \vec{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (15)$$

$$\lambda = -\hbar, \quad \vec{r}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix} \quad (16)$$

The matrix U that diagonalizes G_3 has each eigenvector as a column:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad (17)$$

$$U^\dagger G_3 U = J_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (18)$$

and likewise

$$J_1 = U^\dagger G_1 U \quad (19)$$

$$J_2 = U^\dagger G_2 U \quad (20)$$

Since the G_i and J_i both satisfy the angular momentum commutation relation, they both are representations of the rotation group. They are related to each other by a rotation U .

This finite rotation can be obtained by compounding infinitesimal rotations $\delta\phi$ around axis \hat{n} .

$$\mathbf{V} \rightarrow \mathbf{V}' = \mathbf{V} + \hat{\mathbf{n}}\delta\phi \times \mathbf{V} \quad (21)$$

$$\implies V'_i = [\delta_{ik} + \delta\phi\epsilon_{ijk}n_j] V_k \quad (22)$$

$$= [\delta_{ik} - \delta\phi\epsilon_{ijk}n_i] V_k \quad (23)$$

$$= \left[\delta_{ik} - \delta\phi \frac{i}{\hbar} (G_i)_{jk} n_i \right] V_k \quad (24)$$

$$\implies \mathbf{V}' = \left[\mathbb{1} - \frac{i}{\hbar} \delta\phi \hat{\mathbf{n}} \cdot \mathbf{G} \right] \mathbf{V} \quad (25)$$

So for a finite rotation:

$$\mathbf{V}' = e^{-\frac{i}{\hbar} \delta\phi \hat{\mathbf{n}} \cdot \mathbf{G}} \mathbf{V} \quad (26)$$

2. (a) Let \mathbf{J} be an angular momentum operator. Using the fact that J_x, J_y, J_z satisfy the usual

angular momentum commutation relations, and

$$J_{\pm} \equiv J_x \pm iJ_y, \quad (27)$$

prove

$$J^2 = J_z^2 + J_+J_- - \hbar J_z. \quad (28)$$

$$J_+J_- = (J_x + iJ_y)(J_x - iJ_y) \quad (29)$$

$$= J_x^2 + J_y^2 - iJ_yJ_x - iJ_xJ_y \quad (30)$$

$$= J_x^2 + J_y^2 - i[J_x, J_y] \quad (31)$$

$$= J^2 - J_z^2 + \hbar J_z \quad (32)$$

$$\implies J^2 = J_z^2 + J_+J_- - \hbar J_z \quad (33)$$

(b) Using this, derive the coefficient c_- defined by

$$J_-|jm\rangle = c_-|j, m-1\rangle \quad (34)$$

For $m > -j$ the magnitude is given by

$$\langle jm|J_+J_-|jm\rangle = |c_-|^2 = \langle jm|(J^2 - J_z^2 + \hbar J_z)|jm\rangle \quad (35)$$

$$= (j+m)(j-m+1) \quad (36)$$

The phase is arbitrary, until a convention is chosen for the phases of the states $|jm\rangle$. Normally, these are chosen so that c_- is real and positive:

$$c_- = \sqrt{(j+m)(j-m+1)} \quad (37)$$

3. Consider an orbital angular-momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y -axis. Find the probability for the new state to be found in $m=0, \pm 1$, and ± 2 .

The probability is

$$P(m) = |\langle 2, m | e^{-ih\beta J_y/\hbar} | 2, 0 \rangle|^2 \quad (38)$$

$$= |d_{m,0}^2|^2 \quad (39)$$

We can look up the form of $d_{m,m'}^2$ in a reference, or we can use Wigner's formula.

$$d_{m',m}^j = \sum_k (-1)^{k-m+m'} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(k-m+m')!} \quad (40)$$

$$\times \left(\cos \frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin \frac{\beta}{2}\right)^{2k-m+m'} \quad (41)$$

$$\Rightarrow d_{m,0}^2 = \sum_k (-1)^{k-m} \frac{\sqrt{4(2+m)!(2-m)!}}{(2-k)!k!(2-k-m)!(k+m)!} \left(\cos \frac{\beta}{2}\right)^{4-2k-m} \left(\sin \frac{\beta}{2}\right)^{2k+m}, \quad (42)$$

where terms in the sum are included only when the factorials are not negative.

Note that $d_{m',m}^j = d_{-m',-m}^j$, and so the probabilities with positive and negative m are the same.

So for $m = 2$, we have only the $k = 0$ term

$$d_{2,0}^2 = d_{-2,0}^2 = (-1)^{-2} \frac{\sqrt{4(2+2)!(2-2)!}}{(2)!0!(2-2)!(0+2)!} \left(\cos \frac{\beta}{2}\right)^{4-2} \left(\sin \frac{\beta}{2}\right)^2 \quad (43)$$

$$= \frac{\sqrt{4 * 4!}}{2^2} \frac{1}{4} \sin^2 \beta \quad (44)$$

$$= \frac{\sqrt{6}}{4} \sin^2 \beta \quad (45)$$

For $m = 1$, we have terms with $k = 0$ and $k = 1$.

$$d_{1,0}^2 = -\frac{\sqrt{4 * 3! * 1!}}{2!} \cos^3 \frac{\beta}{2} \sin \frac{\beta}{2} + \frac{\sqrt{24}}{2} \cos \frac{\beta}{2} \sin^3 \frac{\beta}{2} \quad (46)$$

$$= \frac{\sqrt{6}}{2} \sin \beta (\sin^2 \frac{\beta}{2} - \cos^2 \frac{\beta}{2}) \quad (47)$$

$$= \frac{\sqrt{6}}{2} \sin \beta \cos \beta \quad (48)$$

$$= \frac{\sqrt{6}}{4} \sin 2\beta \quad (49)$$

For $m = 0$, we have all 3 terms with $k = 0, 1, 2$.

$$d_{0,0}^2 = \frac{\sqrt{4 * 2! * 2!}}{2! * 2!} \cos^4 \frac{\beta}{2} - \frac{\sqrt{16}}{1} \cos^2 \frac{\beta}{2} \sin^2 \frac{\beta}{2} + \frac{4}{2!2!} \sin^4 \frac{\beta}{2} \quad (50)$$

$$= \cos^4 \frac{\beta}{2} + \sin^4 \frac{\beta}{2} - 4 \cos^2 \frac{\beta}{2} \sin^2 \frac{\beta}{2} \quad (51)$$

$$= \frac{1}{4} (1 + 3 \cos^2 2\beta) \quad (52)$$

So the probability of each of the 5 measurements is

$$P(2) = P(-2) = |d_{\pm 2,0}^2(\beta)|^2 = \frac{3}{8} \sin^4 \beta \quad (53)$$

$$P(1) = P(-1) = |d_{\pm 1,0}^2(\beta)|^2 = \frac{3}{8} \sin^2(2\beta) \quad (54)$$

$$P(0) = |d_{0,0}^2(\beta)|^2 = \frac{1}{16} (3 \cos^2 2\beta + 1)^2 \quad (55)$$

4. A spin-1 particle has the component of its spin in the direction

$$\hat{n} = \frac{1}{\sqrt{3}}(1, 1, 1), \quad (56)$$

measured, and the result is \hbar . Subsequently S_z is measured, with various probabilities of the three possible outcomes.

Let R be a rotation that maps the \hat{z} axis into \hat{n} ,

$$R\hat{z} = \hat{n}. \quad (57)$$

Express the probabilities of the three possible measurement outcomes in terms of the rotation matrix

$$D_{mm'}^1(R) \equiv \langle j = 1, m' | U(R) | j = 1, m \rangle \quad (58)$$

Work out this matrix to find the probabilities explicitly. (You may use tables of d -matrices).

13.2] We have a (spin-1) eigenstate of the spin as measured in the \hat{n} -direction. Call this state $|\hat{n}; m\rangle$

By definition, $S_n |\hat{n}; m\rangle = m |\hat{n}; m\rangle$ ($\hbar=1$)

$$\hat{S}_n = \hat{n} \cdot \vec{S}$$

The spin operators in this system, \vec{S} , are the angular momentum operators? so are subject to the adjoint identity:

$$U(R) \vec{S} U^\dagger(R) = \overleftrightarrow{R} \vec{S} \quad \left(\begin{array}{l} \text{the arrow over the } R \text{ indicates that} \\ \text{it's only meant to act on vectors} \end{array} \right)$$

Let $|m\rangle$ be eigenstates of S_z : $S_z |m\rangle = m |m\rangle$

\hat{R} be a (non-unique) rotation such that $R(\hat{z}) = \hat{n}$

Claim: $|\hat{n}; m\rangle = U(R) |m\rangle$ (up to a phase)

$$\begin{aligned} \text{pf: } S_n |\hat{n}; m\rangle &= S_n (U(R) |m\rangle) = \hat{n} \cdot \vec{S} U(R) |m\rangle \\ &= U(R) U^\dagger(R) \hat{n} \cdot \vec{S} U(R) |m\rangle \\ &= \hat{n} \cdot (U(R) [U^\dagger(R) \vec{S} U(R)]) |m\rangle \end{aligned}$$

$$\begin{aligned} U(R^{-1}) : [U(R)]^\dagger = U^\dagger(R) &\Rightarrow \hat{n} \cdot (U(R) [U(R^{-1}) \vec{S} U^\dagger(R^{-1})]) |m\rangle \\ &= U(R) \hat{n} \cdot (\overleftrightarrow{R} \vec{S}) |m\rangle \end{aligned}$$

$$\vec{a} \cdot \vec{b} = R(\vec{a}) \cdot R(\vec{b}) \quad \Rightarrow \quad = U(R) (R^{-1}(\hat{n}) \cdot \vec{S}) |m\rangle$$

$$\begin{aligned} R(\hat{z}) = \hat{n} \Rightarrow R^{-1}(\hat{n}) = \hat{z} &\Rightarrow = U(R) (\hat{z} \cdot \vec{S}) |m\rangle = U(R) S_z |m\rangle = U(R) m |m\rangle \\ &= m (U(R) |m\rangle) \end{aligned}$$

$\Rightarrow U(R) |m\rangle$ is an eigenvector of S_n ✓

□

- Starting state = $|\hat{n}; +1\rangle$ w/ $\hat{n} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The probability for finding measuring m in a measurement of S_z , then,

$$P_m = |\langle m | \hat{n}_{j+1} \rangle|^2$$

$$\langle m | \hat{n}_{j+1} \rangle = \langle m | U(R) | +1 \rangle = D_{m1}^{\frac{1}{2}, j=1}(R)$$

Side Note: Let R be expressed in Euler angles α, β, γ

$$\Rightarrow U(R) = U_z(\alpha) U_y(\beta) U_z(\gamma)$$

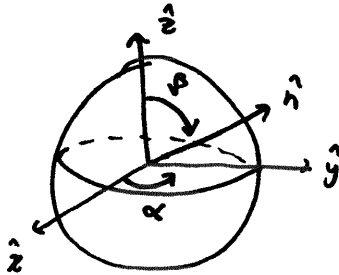
$$\text{Then, } D_{mm'}^j(\alpha, \beta, \gamma) = \langle j m | U(R) | j m' \rangle$$

$$= \langle j m | U_z(\alpha) U_y(\beta) U_z(\gamma) | j m' \rangle$$

$$= e^{im\alpha} \langle j m | U_y(\beta) | j m' \rangle e^{-im'\gamma}$$

$$= e^{i(m\alpha - m'\gamma)} d_{mm'}^j(\beta) \quad (d_{mm'}^j(\beta) = [d_{m'm}^j(\beta)]^*)$$

$$\Rightarrow P_m = |D_{m1}^j(R)|^2 = |e^{i(m\alpha - m'\gamma)} d_{m1}^j(\beta)|^2 = |d_{m1}^j(\beta)|^2 = |d_{1m}^j(\beta)|^2$$



$$\Rightarrow \hat{z} \cdot \hat{n} = \cos \beta$$

$$\Rightarrow P_m = |d_{1m}^1(\cos(\hat{z} \cdot \hat{n}))|^2$$

$$d_{1\pm 1}^1(\beta) = \frac{1 \pm \cos \beta}{2} \Rightarrow P_{\pm 1} = \left| \frac{1 \pm (\hat{z} \cdot \hat{n})}{2} \right|^2 \quad \hat{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \hat{z} \cdot \hat{n} = \frac{1}{\sqrt{3}}$$

$$d_{10}^1(\beta) = \frac{-\sin \beta}{\sqrt{2}} \Rightarrow P_0 = \left| \frac{\sqrt{1 - (\hat{z} \cdot \hat{n})^2}}{\sqrt{2}} \right|^2$$

$$\Rightarrow \boxed{P_{\pm 1} = \frac{1}{3} \pm \frac{\sqrt{3}}{6} \quad P_0 = \frac{1}{3} \quad (P_+ + P_0 + P_-) = 1 \checkmark}$$