Solutions:
Homework Set 6

Due October 8, 2020

1. Find the eigenvalues and eigenvectors of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1}$$

Suppose an electron is in the spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If S_y is measured, what is the probability of the result $+\hbar/2$, in terms of α and β?

The two eigenvalues are $\lambda_\pm = \pm 1$. The corresponding eigenvectors are

$$\psi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{2}$$

$$\psi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{3}$$

The spin operator is $S_y = \hbar/2\sigma_y$, and so has eigenvalues $\pm\hbar/2$.

The probability of measuring $+\hbar/2$ is

$$\frac{1}{\alpha^2+\beta^2} \left| \langle \psi_+ | \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rangle \right|^2 = \frac{1}{\alpha^2+\beta^2} \left| \frac{1}{\sqrt{2}} (i\alpha + \beta) \right|^2 \tag{4}$$

$$= \frac{\alpha^2+\beta^2+2\text{Im}(\alpha\beta^*)}{2(\alpha^2+\beta^2)} \tag{5}$$

$$= \frac{1}{2} - \frac{\text{Im}(\alpha\beta^*)}{\alpha^2+\beta^2} \tag{6}$$

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2. In the 2-dimensional space of a spin-1/2 particle, consider a sequence of Euler rotations represented by

$$U(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right) \tag{7}$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}. \tag{8}$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

In terms of an axis $\hat{n}$ and an angle θ, the rotation operator in spin-1/2 space is (see, e.g., Sakurai 3.2.45)

$$U(\hat{n},\theta) = e^{-\frac{i\theta}{2}\hat{n}\cdot\vec{\sigma}} = \begin{pmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & (-in_x - n_y)\sin\frac{\theta}{2} \\ (-in_x + n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{pmatrix} \tag{9}$$

Setting these equal, one can solve for $(\hat{n},\theta)$ in terms of (α,β,γ), or vice versa. The easiest way to solve for θ is to equate the trace of each matrix

$$\mathrm{Tr}U(\hat{n},\theta) = 2\cos\frac{\theta}{2} \tag{10}$$

$$= \mathrm{Tr}U(\alpha,\beta,\gamma) = 2\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} \tag{11}$$

$$\implies \theta = 2\cos^{-1}\left[2\cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}\right] \tag{12}$$

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3. (a) Consider a pure ensemble of identically prepared spin-1/2 systems. Suppose the expectation values $\langle S_x\rangle$ and $\langle S_z\rangle$ are known, as well as the sign of $\langle S_y\rangle$.

   Show how we may determine the state vector. Why is it unnecessary to know the magnitude of $\langle S_y\rangle$?

A general state in a spin 1/2 system can be written as a superposition of eigenstates of S_z

$$|\psi\rangle = C_+|+\rangle + C_-|-\rangle \tag{13}$$

with

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle \tag{14}$$

$$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle \tag{15}$$

Normalization gives the condition $|C_+|^2 + |C_-|^2 = 1$, and so the state can be written as

$$|\psi\rangle = \cos\frac{\beta}{2}e^{i\phi_+}|+\rangle + \sin\frac{\beta}{2}e^{i\phi_-}|-\rangle \tag{16}$$

for some (real) parameter β. Since the overall phase of a ket is irrelevant, we can multiply by $e^{-i(\phi_+ + \phi_-)/2}$

$$|\psi\rangle = \cos\frac{\beta}{2}e^{i(\phi_+ - \phi_-)/2}|+\rangle + \sin\frac{\beta}{2}e^{-i(\phi_+ - \phi_-)/2}|-\rangle \tag{17}$$

$$= \cos\frac{\beta}{2}e^{i\alpha/2}|+\rangle + \sin\frac{\beta}{2}e^{-i\alpha/2}|-\rangle \tag{18}$$

with $\alpha = (\phi_+ - \phi_-)/2$.

So the state is completely determined by two (real) parameters α and β.

$\langle S_x\rangle$ and $\langle S_z\rangle$ are

$$\langle S_z\rangle = \frac{\hbar}{2}\langle\psi|\big[|+\rangle\langle+| - |-\rangle\langle-|\big]|\psi\rangle \tag{19}$$

$$= \frac{\hbar}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] \tag{20}$$

$$= \frac{\hbar}{2}\cos\beta \tag{21}$$

$$\langle S_x\rangle = \frac{\hbar}{2}\langle\psi|\big[|+\rangle\langle-| + |-\rangle\langle+|\big]|\psi\rangle \tag{22}$$

$$= \frac{\hbar}{2}\left[\cos\frac{\beta}{2}e^{-i\alpha/2}\sin\frac{\beta}{2}e^{-i\alpha/2} + \sin\frac{\beta}{2}e^{i\alpha/2}\cos\frac{\beta}{2}e^{i\alpha/2}\right] \tag{23}$$

$$= \hbar\sin\frac{\beta}{2}\cos\frac{\beta}{2}\cos\alpha \tag{24}$$

$$= \frac{\hbar}{2}\sin\beta\cos\alpha \tag{25}$$

so knowing these, we can solve for β and $\cos\alpha$.

For the average $\langle S_y\rangle$, we have

$$\langle S_y\rangle = -i\frac{\hbar}{2}\langle\psi|\big[|+\rangle\langle-| - |-\rangle\langle+|\big]|\psi\rangle \tag{26}$$

$$= -i\frac{\hbar}{2}\left[\cos\frac{\beta}{2}e^{-i\alpha/2}\sin\frac{\beta}{2}e^{-i\alpha/2} - \sin\frac{\beta}{2}e^{i\alpha/2}\cos\frac{\beta}{2}e^{i\alpha/2}\right] \tag{27}$$

$$= -\frac{\hbar}{2}\sin\beta\sin\alpha. \tag{28}$$

If we already know β and $\cos\alpha$, then we only need go know the sign of $\sin\alpha$, and so we only need the sign of $\langle S_y\rangle$ to completely determine the system.

(b) Consider a mixed ensemble of spin-1/2 systems. Suppose the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$ are all known. Show how we may construct the 2×2 density matrix that characterizes the ensemble.

Again using the S_z basis, we can parameterize the density matrix as

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{29}$$

The ensemble averages are

$$[S_x] = \mathrm{Tr}(\rho S_x) = \frac{\hbar}{2}\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{\hbar}{2}(b+c) \tag{30}$$

$$[S_y] = \frac{\hbar}{2}\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = i\frac{\hbar}{2}(b-c) \tag{31}$$

$$[S_z] = \frac{\hbar}{2}\mathrm{Tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{\hbar}{2}(a-c) \tag{32}$$

We also need the normalization condition for any density matrix

$$\mathrm{Tr}\rho = (a+d) = 1 \tag{33}$$

Solving the 4 equations for (a,b,c,d):

$$a = \langle + | \rho | + \rangle = \frac{1}{\hbar} \left([S_z] + \frac{\hbar}{2} \right) \tag{34}$$

$$b = \langle + | \rho | - \rangle = \frac{1}{\hbar} \left([S_x] - i[S_y] \right) \tag{35}$$

$$c = \langle - | \rho | + \rangle = \frac{1}{\hbar} \left([S_x] + i[S_y] \right) \tag{36}$$

$$d = \langle - | \rho | - \rangle = -\frac{1}{\hbar} \left([S_z] - \frac{\hbar}{2} \right) \tag{37}$$