Solutions:
Homework Set 5

Due September 24

1. As derived in class, the eigenfunctions $\psi_n(x) = \langle x|\psi_n\rangle$ of the harmonic oscillator in configuration space are given by

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{n!2^n}} H_n(\sqrt{\frac{m\omega}{\hbar}}x) e^{-m\omega x^2/2\hbar} \tag{1}$$

with H_n the Hermite polynomials, satisfying the Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}. \tag{2}$$

In this problem, you may use dimensionless units, $m = \omega = \hbar = 1$, as done in class.

Find the corresponding momentum space wave eigenfunctions, $\psi(p) = \langle p|\psi\rangle$.

The system has a symmetry with respect to $x \leftrightarrow p$, which is especially evident in dimensionless units:

$$H = \frac{1}{2}\left(\hat{p}^2 + \hat{x}^2\right), \tag{3}$$

$\Longrightarrow$ Up to a phase α, the wavefunctions are the same as the configuration space wavefunctions:

$$\psi_n(p) = e^{i\alpha_n} \frac{1}{\sqrt{n!2^n}} H_n(p) e^{-p^2/2} \tag{4}$$

The phases are defined in configuration space by Eq. (1), which we must relate to the momentum-space wavefunctions.

The ground state in configuration space (in units $x \leftrightarrow p$) is

$$\psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2} \tag{5}$$

The momentum space ground state is then a Fourier transform:

$$\psi_0(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \frac{1}{\pi^{1/4}} e^{-x^2/2} = \frac{1}{\pi^{1/4}} e^{-p^2/2} \tag{6}$$

So the phase $\alpha_0 = 0$. To get the phase of the other states we note

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle, \tag{7}$$

$$a^\dagger = \frac{x - ip}{\sqrt{2}} = \frac{-i}{\sqrt{2}} \left(p - \frac{d}{dp} \right), \tag{8}$$

$$\Longrightarrow \psi_n(p) = \frac{(-i)^n}{\pi^{1/4}} \frac{1}{\sqrt{n! 2^n}} \left(p - \frac{d}{dp} \right)^n e^{-p^2/2} \tag{9}$$

$$\tag{10}$$

Comparing to the derivation of the configuration space wavefunction, we can see that $e^{i\alpha_n} = (-i)^n$, and

$$\psi_n(p) = \frac{1}{\pi^{1/4}} \frac{(-i)^n}{\sqrt{n! 2^n}} H_n(p) e^{-p^2/2} \tag{11}$$

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2. In classical mechanics, any two Lagrangians that differ by a total time derivative produce the same equations of motion. For example, in one dimension, L and L', defined by

$$L' = L + \frac{df(x,t)}{dt} = L + \frac{\partial f}{\partial t} + \dot{x} \frac{\partial f}{\partial x} \tag{12}$$

give the same equations of motion. This is easily verified by using both L and L' in the Euler-Lagrange equations,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \tag{13}$$

(a) Let L_0 be the classical Lagrangian for a free particle

$$L_0(x, \dot{x}) = \frac{m}{2}\dot{x}^2, \tag{14}$$

and L_S be the classical Lagrangian for a particle in a uniform gravitational field (with x-axis pointing up),

$$L_s(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - mgx \tag{15}$$

According to the principle of equivalence, motion in an accelerated frame is physically indistinguishable from motion in a uniform gravitational field.

Consider a region of space free of gravitational fields, where the particle motion in an inertial frame with coordinate x is described by Lagrangian $L_0(x, \dot{x})$. Let y be the coordinate in a frame that is accelerated at a constant acceleration g in the $+x$ direction. Assume that the origins of the inertial frame (x) and accelerated frame (y) coinside at $t = 0$.

Transform $L_0(x, \dot{x})$ to the y coordinate, and show that the result is $L_S(y, \dot{y})$ plus the exact time derivative of a function $f(y, t)$:

$$L_0(x, \dot{x}) = L_s(y, \dot{y}) + \frac{d}{dt}f(y, t) \tag{16}$$

Determine $f(y, t)$.

$$L_0(x, \dot{x}) = \frac{m}{2}\dot{x}^2 \tag{17}$$

$$x = y + \frac{1}{2}gt^2 \tag{18}$$

$$\dot{x} = \dot{y} + gt \tag{19}$$

$$\implies L_0(x, \dot{x}) = \frac{m}{2}(\dot{y} + gt)^2 \tag{20}$$

$$= \frac{m}{2}(\dot{y}^2 + 2\dot{y}gt + g^2t^2) \tag{21}$$

$$= \frac{m}{2}\dot{y}^2 - mgy + mgy + m\dot{y}gt + \frac{m}{2}g^2t^2 \tag{22}$$

$$= L_g(y, \dot{y}) + \frac{d}{dt}\left(mgyt + \frac{m}{6}g^2t^3\right) \tag{23}$$

$$f(y, t) = mgyt\frac{m}{6}g^2t^3 \tag{24}$$

(b) Let H_0 and H_g be the quantum Hamiltonians for a free particle and a particle in a uniform gravitational field,

$$H_0 = \frac{p^2}{2m}, \quad H_S = \frac{p^2}{2m} + mgx, \tag{25}$$

and let $U_0(t)$ and $U_g(t)$ be corresponding time evolution operators,

$$U_0(t) = e^{-iH_0 t/\hbar}, \quad U_g(t) = e^{-iH_g t/\hbar}. \tag{26}$$

The propagator for the free particle is

$$\langle x_1 | U_0(t) | x_0 \rangle = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left[\frac{i}{\hbar}\frac{m(x_1 - x_0)^2}{2t}\right]. \tag{27}$$

Use the path integral to find the propagator of a particle in a uniform gravitational field, $\langle x_1 | U_g(t) | x_0 \rangle$.

Hint: you do not need the detailed, discretized version of the path integral; instead, just use the compact form

$$\langle x_1 | U(t) | x_0 \rangle = \int d\left[x(\tau)\right] \exp\left(\frac{i}{\hbar}\int_0^t L d\tau\right) \tag{28}$$

and follow the obvious rules of calculus in manipulating it.

$$\begin{aligned}
\langle x_1 | U_0(t) | x_0 \rangle &= \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left[\frac{i}{\hbar}\frac{m(x_1 - x_0)^2}{2t}\right] && (29)\\
&= \int d\left[x(\tau)\right] \exp\left(\frac{i}{\hbar}\int_0^t d\tau L_0(x, \dot{x})\right) && (30)\\
&= \int d\left[y(\tau)\right] \exp\left(\frac{i}{\hbar}\int_0^t d\tau L_s(y, \dot{y}) + \frac{d}{dt} f(y, t)\right) && (31)
\end{aligned}$$

Since $y(\tau) = x(\tau) + \frac{1}{2} g\tau^2$,
$\implies d[y(\tau)] = d[x(\tau)]$.

The paths satisfy

$$y(0) = x(0) = x_0 \tag{32}$$
$$y(t) = x(t) - \frac{1}{2}gt^2 = y_1 = x_1 - \frac{1}{2}gt^2 \tag{33}$$

So the path integral is

$$\langle x_1|U_0(t)|x_0\rangle = \int d\left[y(\tau)\right] \exp\left(\frac{i}{\hbar}\int_0^t d\tau L_s(y,\dot{y}) + \frac{i}{\hbar}\left[f(y_1,t) - f(y_0,0)\right]\right) \tag{34}$$

with

$$f(y_1,t) - f(y_0,0) = mgy_1 t + \frac{m}{6}g^2t^3. \tag{35}$$

This is independent of the path $y(\tau)$, and can be taken out of the path integral

$$\langle x_1|U_0(t)|x_0\rangle = e^{\frac{i}{\hbar}\left(mgy_1t+\frac{m}{6}g^2t^3\right)} \int d\left[y(\tau)\right] \exp\left(\frac{i}{\hbar}\int_0^t d\tau L_s(y,\dot{y})\right) \tag{36}$$
$$= e^{\frac{i}{\hbar}\left(mgy_1t+\frac{m}{6}g^2t^3\right)}\langle y_1|U_g(t)|y_0\rangle \tag{37}$$

So, finally,

$$\langle y_1|U_g(t)|y_0\rangle = e^{-\frac{i}{\hbar}\left(mgy_1t+\frac{m}{6}g^2t^3\right)}\langle x_1|U_0(t)|x_0\rangle \tag{38}$$
$$= e^{-\frac{i}{\hbar}\left(mgy_1t+\frac{m}{6}g^2t^3\right)}\sqrt{\frac{m}{2\pi i\hbar t}}e^{\frac{i}{\hbar}\frac{m\left[y_1+\frac{1}{2}gt^2-y_0\right]^2}{2t}} \tag{39}$$

Or, taking $y \to x$,

$$\langle x_1|U_g(t)|x_0\rangle = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\frac{i}{\hbar}\left[\frac{m(x_1-x_0)}{2t} - \frac{m}{2}gt(x_1+x_0) - \frac{mg^2t^3}{24}\right] \tag{40}$$