Homework Set 5

Due September 24

1. As derived in class, the eigenfunctions $\psi_n(x) = \langle x|\psi_n\rangle$ of the harmonic oscillator in configuration space are given by

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{n!2^n}} H_n(\sqrt{\frac{m\omega}{\hbar}}x) e^{-m\omega x^2/2\hbar} \tag{1}$$

with H_n the Hermite polynomials, satisfying the Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}. \tag{2}$$

In this problem, you may use dimensionless units, $m = \omega = \hbar = 1$, as done in class.

Find the corresponding momentum space wave eigenfunctions, $\psi(p) = \langle p|\psi\rangle$.

2. In classical mechanics, any two Lagrangians that differ by a total time derivative produce the same equations of motion. For example, in one dimension, L and L', defined by

$$L' = L + \frac{df(x,t)}{dt} = L + \frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} \tag{3}$$

give the same equations of motion. This is easily verified by using both L and L' in the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \tag{4}$$

(a) Let L_0 be the classical Lagrangian for a free particle

$$L_0(x, \dot{x}) = \frac{m}{2}\dot{x}^2, \tag{5}$$

and L_S be the classical Lagrangian for a particle in a uniform gravitational field (with x-axis pointing up),

$$L_s(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - mgx \tag{6}$$

According to the principle of equivalence, motion in an accelerated frame is physically indistinguishable from motion in a uniform gravitational field.

Consider a region of space free of gravitational fields, where the particle motion in an inertial frame with coordinate x is described by Lagrangian $L_0(x, \dot{x})$. Let y be the coordinate in a frame that is accelerated at a constant acceleration g in the $+x$ direction. Assume that the origins of the inertial frame (x) and accelerated frame (y) coinside at $t = 0$.

Transform $L_0(x, \dot{x})$ to the y coordinate, and show that the result is $L_S(y, \dot{y})$ plus the exact time derivative of a function $f(y, t)$:

$$L_0(x, \dot{x}) = L_s(y, \dot{y}) + \frac{d}{dt}f(y, t) \tag{7}$$

Determine $f(y, t)$.

(b) Let H_0 and H_g be the quantum Hamiltonians for a free particle and a particle in a uniform gravitational field,

$$H_0 = \frac{p^2}{2m}, \quad H_S = \frac{p^2}{2m} + mgx, \tag{8}$$

and let $U_0(t)$ and $U_g(t)$ be corresponding time evolution operators,

$$U_0(t) = e^{-iH_0t/\hbar}, \quad U_g(t) = e^{-iH_gt/\hbar}. \tag{9}$$

The propagator for the free particle is

$$\langle x_1|U_0(t)|x_0\rangle = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[\frac{i}{\hbar}\frac{m(x_1 - x_0)^2}{2t}\right]. \tag{10}$$

Use the path integral to find the propagator of a particle in a uniform gravitational field, $\langle x_1|U_g(t)|x_0\rangle$.

Hint: you do not need the detailed, discretized version of the path integral; instead, just use the compact form

$$\langle x_1|U(t)|x_0\rangle = \int d\left[x(\tau)\right] \exp\left(\frac{i}{\hbar}\int_0^t L d\tau\right) \tag{11}$$

and follow the obvious rules of calculus in manipulating it.