Homework Set 4

Due September 17, 2020

1. In this problem we consider the spreading of wave packets of a free particle in one dimension. The Hamiltonian is

$$H = \frac{p^2}{2m} \tag{1}$$

   (a) Let the wave function at time $t = 0$ be

$$\psi_0(x) = \psi(x, t = 0) = Ce^{-x^2/(4L^2)}, \tag{2}$$

   where C and L are constants (with L real).

   Normalize the wave function and find C, choosing C real and positive, $C > 0$.

   Then determine $\langle x \rangle$, $\langle p \rangle$ and $\Delta x \equiv \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ at $t = 0$.

   (b) Compute the initial momentum space wave function $\psi_0(p)$, and from that compute $\Delta p \equiv \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$.

   Compare $\Delta x \Delta p$ to the minimum value allowed by the uncertainty principle

   (c) Now compute $\psi(p, t)$ for any time t. Use it to find $\langle p \rangle$ and Δp for any time t.

   (d) Now compute $\psi(x, t)$ for any time t. Express your answer in terms of the quantity,

$$D = L + \frac{it\hbar}{2mL}, \tag{3}$$

   (a useful abbreviation). Then find $\langle x \rangle$ and Δx for any time t. Notice what happens to the product $\Delta x \Delta p$ for times $t > 0$.

   (e) If you had not done a detailed calculation you could estimate Δx as a function of t by using the uncertainty principle. Do this and compare to the results of the detailed calculation

2. Consider a single, spinless particle moving in one dimension, confined by rigid walls at $x = 0$ and $x = a$.

   That is, a particle in an infinite square well

$$H = \frac{p^2}{2m} + V(x), \tag{4}$$

$$V(x) = \begin{cases} \infty & x > a \\ 0 & 0 \leq x \leq a \ , \\ \infty & x < 0 \end{cases} \tag{5}$$

   so that the wavefunction satisfies the free particle Schrödinger equation for $0 \leq x \leq a$, but the probability of measuring the particle with position $x < 0$ or $x > a$ is zero.

   (a) Find an expression for the energy eigenstates $\psi_n(x)$ and eigenvalues E_n. Is there any degeneracy?

   (b) For each energy eigenstate, evaluate

$$\Delta x^2 \Delta p^2 \equiv \left(\langle x^2 \rangle - \langle x \rangle^2\right)\left(\langle p^2 \rangle - \langle p \rangle^2\right) \tag{6}$$

   (c) For which energy eigenstate is this the smallest? How does it compare to the minimum allowed by the uncertainty principle?