

Solutions:

Homework Set 3

Due September 10, 2020

1. The projection postulate of quantum mechanics says that if a system is described by a pure state $|\psi\rangle$ (here assumed to be normalized), then after a measurement of the operator A producing eigenvalue a_n , the system is described by the (normalized) state

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}, \quad (1)$$

where P_n is the projector onto the n -th eigenspace of A .

I.e., for no degeneracy

$$P_n = |n\rangle\langle n|. \quad (2)$$

When there is a degeneracy of order M , we define an orthonormal basis within the eigenspace, labeled by j , and

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Suppose instead the system is a mixed state described by a density operator ρ (assumed normalized). What is the (normalized) density operator ρ' after a measurement with result a_n ? Express your answer in terms of the original density operator ρ . Do not assume the eigenvalue a_n is nondegenerate.

Let ρ be the original density matrix, expressed in terms of pure states and probabilities by

$$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i| \quad (5)$$

Suppose we have a large number N of systems. Then

$$N_i = N f_i \quad (6)$$

is the number in state $|\psi_i\rangle$.

The number of the N_i systems that have measurement result a_n are

$$N'_i = N_i \langle\psi_i|P_n|\psi_i\rangle. \quad (7)$$

The state of the systems that were in $|\psi_i\rangle$ becomes

$$|\psi'_i\rangle = \frac{P_n|\psi_i\rangle}{\sqrt{\langle\psi_i|P_n|\psi_i\rangle}}, \quad (8)$$

Therefore the total number of systems after the measurement is

$$N' = \sum_i N'_i = \sum_i N_i \langle\psi_i|P_n|\psi_i\rangle. \quad (9)$$

So the probability of being in state $|\psi'_i\rangle$ after the measurement is

$$f'_i = \frac{N'_i}{N'}. \quad (10)$$

But

$$N' = \sum_i N_i \langle\psi_i|P_n|\psi_i\rangle = N \sum_i f_i \langle\psi_i|P_n|\psi_i\rangle = N \text{Tr}(\rho P_n). \quad (11)$$

So

$$\rho' = \sum_i f'_i |\psi'_i\rangle\langle\psi'_i| \quad (12)$$

$$= \sum_i \frac{N'_i}{N'} \frac{P_n |\psi_i\rangle\langle\psi_i| P_n}{\langle\psi_i|P_n|\psi_i\rangle} \quad (13)$$

$$= \sum_i \frac{N_i \langle\psi_i|P_n|\psi_i\rangle}{N'} \frac{P_n |\psi_i\rangle\langle\psi_i| P_n}{i \langle\psi_i|P_n|\psi_i\rangle} \quad (14)$$

$$= \frac{N}{N \text{Tr}(\rho P_n)} \sum_i f_i P_n |\psi_i\rangle\langle\psi_i| P_n \quad (15)$$

$$= \frac{P_n \rho P_n}{\text{Tr}(\rho P_n)} \quad (16)$$

2. Find the linear combination of eigenkets of the S_z operator, $|+\rangle$ and $|-\rangle$, that maximize the uncertainty in $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle$. Show that for the combination you find that the uncertainty relation is not violated.

In terms of $|+\rangle$ and $|-\rangle$, S_x and S_y can be written

$$S_x = \frac{\hbar}{2} (|-\rangle\langle+| + |+\rangle\langle-|) \quad (17)$$

$$S_y = i\frac{\hbar}{2} (|-\rangle\langle+| - |+\rangle\langle-|) \quad (18)$$

Since we can choose the overall phase, an arbitrary normalized state can be written as

$$|\alpha\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)e^{i\delta}|-\rangle \quad (19)$$

Then

$$\langle S_x \rangle = \hbar \cos(\beta/2) \sin(\beta/2) \cos \delta \quad (20)$$

$$\langle S_y \rangle = \hbar \cos(\beta/2) \sin(\beta/2) \sin \delta \quad (21)$$

$$\langle S_x^2 \rangle = \frac{\hbar^2}{4} \quad (22)$$

$$\langle S_y^2 \rangle = \frac{\hbar^2}{4} \quad (23)$$

$$\langle (\Delta S_x)^2 \rangle = \frac{\hbar^2}{4} - \hbar^2 \cos^2(\beta/2) \sin^2(\beta/2) \cos^2 \delta \quad (24)$$

$$\langle (\Delta S_y)^2 \rangle = \frac{\hbar^2}{4} - \hbar^2 \cos^2(\beta/2) \sin^2(\beta/2) \sin^2 \delta \quad (25)$$

So we have

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = \hbar^4 \left[\frac{1}{16} - \frac{1}{4} \cos^2(\beta/2) \sin^2(\beta/2) + \cos^4(\beta/2) \sin^4(\beta/2) \cos^2 \delta \sin^2 \delta \right] \quad (26)$$

$$= \hbar^4 \left[\frac{1}{16} - \frac{1}{4} \sin^2(\beta) + \frac{1}{4} \sin^4(\beta) \sin^2(2\delta) \right] \quad (27)$$

The last term is positive definite, and maximal at $\delta = \frac{\pi}{4}$, or $\sin(2\delta) = 1$.

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = \hbar^4 \left[\frac{1}{16} - \frac{1}{4} \sin^2 \beta + \frac{1}{4} \sin^4 \beta \right] \quad (28)$$

$$= \left[\hbar^2 \left(\frac{1}{4} - \sin^2 \beta \right) \right]^2 \quad (29)$$

The maximum occurs when $\sin \beta = 0 \implies \beta = 0, \pi$

I.e., when $|\alpha\rangle$ is one of the eigenstates of S_z

The uncertainty relation then states

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16} \quad (30)$$

$$\geq \frac{1}{4} \langle[S_x, S_y]\rangle^2 \quad (31)$$

$$= \frac{1}{4} \langle\hbar S_z\rangle^2 \quad (32)$$

$$= \frac{1}{4} \left(\frac{\hbar^2}{2}\right)^2 \quad (33)$$

The inequality holds (and is saturated).

3. A three-dimensional ket space is spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, and $|3\rangle$. In this basis, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}. \quad (34)$$

- (a) Do A or B have degenerate eigenvalues?
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By inspection, the eigenvalues of A are a , $-a$, and $-a$, and there is a degeneracy.

B can be written as

$$B = \left(\begin{array}{c|c} b & \\ \hline & b\sigma_2 \end{array} \right), \quad (35)$$

with σ_2 a Pauli matrix, which has eigenvalues of ± 1 .

$\implies B$ has eigenvalues b , b , and $-b$, and is also degenerate.

- (b) Show that A and B commute.
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One can multiply the matrices explicitly, or note the following:

A and B can be written as

$$A = a \left(\begin{array}{c|c} \mathbb{1}_1 & \\ \hline & -\mathbb{1}_2 \end{array} \right), \quad B = b \left(\begin{array}{c|c} \mathbb{1}_1 & \\ \hline & \sigma_2 \end{array} \right) \quad (36)$$

where $\mathbb{1}_1$ and $\mathbb{1}_2$ are 1- and 2-dimensional identity matrices, respectively.

$$\implies [A, B] = ab \left(\frac{[\mathbb{1}_1, \mathbb{1}_1]}{[-\mathbb{1}_2, \sigma_2]} \right) = 0 \quad (37)$$

- (c) Find a new set of kets that are the simultaneous eigenkets of A and B . Specify the eigenvalues of A and B for each ket. Are these eigenvalues sufficient to uniquely specify each ket?

The eigenvector of A corresponding to the nondegenerate eigenvalue of A is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle \quad (38)$$

$|1\rangle$ is also an eigenvector of B , so we choose the first vector in our simultaneous basis:

$$|1'\rangle = |1\rangle \quad (39)$$

Any vectors in the remaining subspace will automatically be eigenkets of A . We want to choose the two orthogonal vectors within this space that are eigenvectors of B , and we will then have simultaneous eigenvectors.

\implies We need to diagonalize σ_2 within the $|2\rangle, |3\rangle$ subspace. \implies we should choose

$$|2'\rangle = \frac{1}{\sqrt{2}}(|2\rangle + i|3\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} \quad (40)$$

$$|3'\rangle = \frac{1}{\sqrt{2}}(|2\rangle - i|3\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix} \quad (41)$$

These are now simultaneous eigenvectors of A and B , with eigenvalues

$$|1'\rangle = |a, b\rangle \quad (42)$$

$$|2'\rangle = |-a, b\rangle \quad (43)$$

$$|3'\rangle = |-a, -a\rangle \quad (44)$$

The degeneracy is completely broken, and the set of eigenvalues of A and B , is sufficient to specify each state.

4. Let $f(\mathbf{x})$ be an arbitrary function of \mathbf{x} and $g(\mathbf{p})$ an arbitrary function of \mathbf{p} . Evaluate the commutators, $[p_i, f(\mathbf{x})]$ and $[x_i, g(\mathbf{p})]$

$$4.4. \quad [p_i, f(\vec{x})] = p_i f(\vec{x}) - f(\vec{x}) p_i$$

$$\begin{array}{l} \psi(\vec{x}) \xrightarrow{f(\vec{x})} f(\vec{x})\psi(\vec{x}) \xrightarrow{p_i = -i\hbar \frac{\partial}{\partial x_i}} -i\hbar \left[\frac{\partial f}{\partial x_i} \psi + f \frac{\partial \psi}{\partial x_i} \right] \\ \downarrow p_i \quad \quad \quad \downarrow f(\vec{x}) \\ p_i \psi(\vec{x}) \xrightarrow{p_i} -i\hbar \frac{\partial \psi}{\partial x_i} \xrightarrow{f(\vec{x})} -i\hbar f(\vec{x}) \frac{\partial \psi}{\partial x_i} \end{array}$$

Subst, $-i\hbar \frac{\partial f}{\partial x_i} \psi$

$$\Rightarrow \boxed{[p_i, f(\vec{x})] = -i\hbar \frac{\partial f}{\partial x_i}}$$

For other commutator use momentum wave fns.

$$\begin{array}{l} \phi(\vec{p}) \xrightarrow{g(\vec{p})} g(\vec{p})\phi(\vec{p}) \xrightarrow{x_i = i\hbar \frac{\partial}{\partial p_i}} i\hbar \left[\frac{\partial g}{\partial p_i} \phi + g \frac{\partial \phi}{\partial p_i} \right] \\ \downarrow x_i \quad \quad \quad \downarrow g(\vec{p}) \\ x_i \phi(\vec{p}) \xrightarrow{x_i} i\hbar \frac{\partial \phi}{\partial p_i} \xrightarrow{g(\vec{p})} i\hbar g(\vec{p}) \frac{\partial \phi}{\partial p_i} \end{array}$$

Subst, get $i\hbar \frac{\partial g}{\partial p_i} \phi$

$$\Rightarrow \boxed{[x_i, g(\vec{p})] = i\hbar \frac{\partial g}{\partial p_i}}$$