Solutions:
Homework Set 2

Due August 31, 2020

1. The problem of finding the eigenkets and eigenbras of an arbitrary operator is more complicated and full of exceptions than in the case of Hermitian operators. There are, however, other classes of operators that share many of the nice properties of Hermitian operators. These include anti-Hermitian and unitary operators.

   We define an operator to be *normal* if it commutes with its Hermitian conjugate, $[A, A^\dagger] = 0$. Notice that Hermitian, anti-Hermitian, and unitary operators are normal. In the following you may assume that you are working in a finite-dimensional Hilbert space.

   (a) Show that if A is normal, and $A|u\rangle = a|u\rangle$ for some nonzero $|u\rangle$, then $A^\dagger|u\rangle = a^*|u\rangle$. Thus, the eigenbras of A are the Hermitian conjugates of the eigenkets, and the left spectrum is identical to the right spectrum. Hint: it is not necessary to introduce orthonormal bases or anything similar.

Suppose $[A, A^\dagger] = 0$ and $A|u\rangle = a|u\rangle$ ($\implies \langle u|A^\dagger = \langle u|a^*$).

Let

$$|\phi\rangle \equiv (A^\dagger - a^*)|u\rangle, \tag{1}$$

Consider

$$\begin{aligned} \langle\phi|\phi\rangle &= \langle u|(A - a)(A^\dagger - a^*)|u\rangle &(2)\\ &= \langle u| \left(AA^\dagger - Aa^* - A^\dagger a + |a|^2\right) |u\rangle &(3)\\ &= \langle u| \left(|a|^2 - |a|^2 - |a|^2 + |a|^2\right) |u\rangle = 0 &(4) \end{aligned}$$

$\implies |\phi\rangle = |0\rangle$, or equivalently $A^\dagger|u\rangle = a^*|u\rangle$

(b) Show that the eigenspaces corresponding to distinct eigenvalues of a normal operator are orthogonal. This is a generalization of the easy and familiar proof for Hermitian operators.

Let

$$A|u_1\rangle = a_1|u_1\rangle \tag{5}$$
$$A|u_2\rangle = a_2|u_2\rangle \tag{6}$$

Eq. (5) implies

$$\langle u_2|A|u_1\rangle = a_1\langle u_2|u_1\rangle \tag{7}$$

From the results of part 1a and Eq. (6):

$$A^\dagger|u_2\rangle = a_2^*|u_2\rangle \tag{8}$$
$$\implies \langle u_2|A = a_2\langle u_2| \tag{9}$$
$$\implies \langle u_2|A|u_1\rangle = a_2\langle u_2|u_1\rangle \tag{10}$$

Subtracting Eq. (7) from (10) gives

$$(a_1 - a_2)\langle u_2|u_1\rangle = 0 \tag{11}$$

$\implies$ either $a_1 = a_2$, or $\langle u_2|u_1\rangle = 0$.

---

2. If the kets $|\alpha\rangle$ and $|\beta\rangle$ are eigenkets of the Hermitian operator $\hat{A}$, under what conditions will the superposition $|\psi\rangle = |\alpha\rangle + |\beta\rangle$ be an eigenstate of $\hat{A}$? Explain.

Denote the eigenvalues α and β, respectively:

$$\hat{A}|\alpha\rangle = \alpha|\alpha\rangle \tag{12}$$
$$\hat{A}|\beta\rangle = \beta|\beta\rangle \tag{13}$$

$$\hat{A}|\psi\rangle = \alpha|\alpha\rangle + \beta|\beta\rangle \tag{14}$$

$$\stackrel{?}{=} \gamma|\psi\rangle = \gamma\left(|\alpha\rangle + |\beta\rangle\right) \tag{15}$$

This is only possible when the eigenvalues are degenerate $\alpha = \beta$

---

3. Consider an N-dimensional ket space spanned by the N eigenkets $\{|a_i\rangle\}$ (where $i = 1, \ldots, N$) of a Hermitian operator $\hat{A}$. Assume that the eigenvalues a_i are nondegenerate.

   (a) What ket(s) are produced by the action of the operator

$$\hat{\theta} \equiv \prod_{i=1}^{N} \left(\hat{A} - a_i\right) = (\hat{A} - a_1)(\hat{A} - a_2)(\hat{A} - a_3)\ldots(\hat{A} - a_N) \tag{16}$$

   on ANY arbitrary ket in the ket space?

The eigenkets of $\hat{A}$ span the space, and any ket can be written as a superposition of them:

$$|\psi\rangle = \sum_{i-1}^{N} c_i|a_i\rangle \tag{17}$$

with

$$c_i = \langle a_i|\psi\rangle. \tag{18}$$

$\hat{\theta}$ operating on an arbitrary state gives:

$$\hat{\theta}|\psi\rangle = \sum_{j-1}^{N} c_j \prod_{i=1}^{N}(\hat{A} - a_i)|a_j\rangle \tag{19}$$

$$= \sum_{j-1}^{N} c_j \prod_{i=1}^{N}(a_j - a_i)|a_j\rangle \tag{20}$$

For every term in the sum, there will be one factor with $i = j$ which is zero, and the final result is the null ket.

$$\hat{\theta}|\psi\rangle = |0\rangle \tag{21}$$

Since this is true for any ket, $\hat{\theta}$ is the null operator

---

(b) Setting $\hat{A} = \hat{S}_z$ and acting on the ket space associated with a spin-$\frac{1}{2}$ particle, demonstrate your answer to part (a).

Using the eigenbasis of S_x:

$$S_x|+\rangle = \frac{\hbar}{2}|+\rangle \tag{22}$$

$$S_x|-\rangle = -\frac{\hbar}{2}|+\rangle \tag{23}$$

$$\implies \hat{\theta} = (S_x - \frac{\hbar}{2})(S_x + \frac{\hbar}{2}) \tag{24}$$

Consider an arbitrary ket, written in the eigenbasis

$$|\psi\rangle = a_+|+\rangle + a_-|-\rangle \tag{25}$$

$$\hat{\theta}|\psi\rangle = (S_x - \frac{\hbar}{2})(S_x + \frac{\hbar}{2})\left(a_+|+\rangle + a_-|-\rangle\right) \tag{26}$$

$$= \frac{\hbar}{2}\left[a_+(1-1)(1+1)|+\rangle + a_-(1+1)(1-1)|-\rangle\right] \tag{27}$$

$$= 0 \tag{28}$$

(c) What is the commutator of θ with itself?

$$[\hat{\theta}, \hat{\theta}] = 0 \tag{29}$$