Solutions:
Homework Set 1

Due August 24, 2020

1. Consider the 2x2 matrices:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

The three matrices σ_x, σ_y, and σ_z are called the *Pauli matrices*, which can also be denoted $\sigma_1, \sigma_2, \sigma_3$, respectively

(a) Show that

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{2}$$

with $i \in \{1, 2, 3\}$. Here δ_{ij} is understood to be multiplied by $\mathbb{1}$, we use the summation convention (repeated indices k are summed) and ϵ_{ijk} is the completely antisymmetric Levi-Civita symbol.

A simple way to show it is to directly multiply each pair of matrices to see

$$\sigma_1\sigma_1 = \sigma_2\sigma_2 = \sigma_3\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \tag{3}$$

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 \tag{4}$$

$$\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2 \tag{5}$$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1 \tag{6}$$

etc.

(b) Consider a vector operator — that is, a set of operators $\boldsymbol{A} = (A_x, A_y, A_z)$, where A_x, A_y, A_x are operators.

Given two vector operators $\boldsymbol{A}, \boldsymbol{B}$ that commute with $\boldsymbol{\sigma}$ but not each other, use Eq. (2) to show that

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{B} + i\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}), \tag{7}$$

where

$$\boldsymbol{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}, \tag{8}$$

and two vector operators commute if all their components commute.

Note that in general, $\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$, and $\boldsymbol{A} \times \boldsymbol{B} \neq -\boldsymbol{B} \times \boldsymbol{A}$

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) &= \sigma_i A_i \sigma_j B_j = \sigma_i \sigma_j A_i B_j & (9)\\
&= (\delta_{ij} + i\epsilon_{ijk}\sigma_k) A_i B_j & (10)\\
&= A_i B_i + i\sigma_k(\epsilon_{ijk} A_i B_j) & (11)\\
&= \boldsymbol{A} \cdot \boldsymbol{B} + i\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}), & (12)
\end{aligned}$$

(c) Let $\hat{n}$ be an arbitrary unit vector and θ an arbitrary angle. Show that

$$\exp(-i\theta\boldsymbol{\sigma} \cdot \hat{n}) = \mathbb{1}\cos\theta - i(\boldsymbol{\sigma} \cdot \hat{n})\sin\theta \tag{13}$$

An exponential of a matrix (or any operator) is defined by its Taylor series

$$\exp(-i\theta\boldsymbol{\sigma} \cdot \hat{n}) = \mathbb{1} - i\theta(\boldsymbol{\sigma} \cdot \hat{n}) - \frac{\theta^2}{2}(\boldsymbol{\sigma} \cdot \hat{n})(\boldsymbol{\sigma} \cdot \hat{n}) + \frac{i\theta^3}{3!}(\boldsymbol{\sigma} \cdot \hat{n})^3 + \dots \tag{14}$$

From the previous result we know that

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \hat{n})(\boldsymbol{\sigma} \cdot \hat{n}) &= \hat{n} \cdot \hat{n} + i\boldsymbol{\sigma} \cdot (\hat{n} \times \hat{n}) & (15)\\
&= 1 + 0 & (16)\\
&= \mathbb{1} & (17)
\end{aligned}$$

So

$$(\boldsymbol{\sigma} \cdot \hat{n})^3 = (\boldsymbol{\sigma} \cdot \hat{n}) \tag{18}$$

$$(\boldsymbol{\sigma} \cdot \hat{n})^4 = \mathbb{1} \tag{19}$$

$$(\boldsymbol{\sigma} \cdot \hat{n})^5 = (\boldsymbol{\sigma} \cdot \hat{n}) \tag{20}$$

$$\vdots \tag{21}$$

or

$$(\boldsymbol{\sigma} \cdot \hat{n})^{2m} = \mathbb{1}, \quad m \in \{0, 1, 2, 3, \ldots\} \tag{22}$$

$$(\boldsymbol{\sigma} \cdot \hat{n})^{2m+1} = (\boldsymbol{\sigma} \cdot \hat{n}) \tag{23}$$

So the exponential is

$$\exp(-i\theta\boldsymbol{\sigma} \cdot \hat{n}) = \mathbb{1}\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) - (\boldsymbol{\sigma} \cdot \hat{n})\left(\theta - \frac{\theta^3}{3!} + \ldots\right) \tag{24}$$

$$= \mathbb{1}\cos\theta - i(\boldsymbol{\sigma} \cdot \hat{n})\sin\theta \tag{25}$$

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2. (a) Consider two operators A, B that do not commute. Show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots \tag{26}$$

Hint: Replace A by λA, where λ is a parameter, and let the left-hand side be $F(\lambda)$. Find a differential equation satisfied by $F(\lambda)$ and solve it. Alternatively, expand $F(\lambda)$ in a Taylor series in λ. At the end, set $\lambda = 1$.

Let $F(\lambda) \equiv e^{\lambda A} B e^{-\lambda A}$. Then

$$\begin{aligned}
\frac{dF}{d\lambda} &= e^{\lambda A} A B e^{-\lambda A} - e^{\lambda A} B A e^{-\lambda A} && (27)\\
&= e^{\lambda A}(AB - BA)e^{-\lambda A} && (28)\\
&= e^{\lambda A}[A, B]e^{-\lambda A} && (29)\\
\frac{dF^2}{d\lambda^2} &= e^{\lambda A}[A, [A, B]]e^{-\lambda A} && (30)\\
\frac{dF^3}{d\lambda^3} &= e^{\lambda A}[A, [A, [A, B]]]e^{-\lambda A} && (31)\\
&\vdots && (32)\\
\left.\frac{dF^n}{d\lambda^n}\right|_{\lambda=0} &= \underbrace{[A, [A, [A, \ldots,}_{n \text{ times}} B]] \ldots] && (33)
\end{aligned}$$

So, finally, we can write the Taylor series of F around $\lambda = 0$ as

$$\begin{aligned}
F(\lambda) &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left.\frac{dF^n}{d\lambda^n}\right|_{\lambda=0} && (34)\\
&= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \underbrace{[A, [A, [A, \ldots,}_{n \text{ times}} B]] \ldots] && (35)\\
&= B + \lambda[A, B] + \frac{\lambda^2}{2!}[A, [A, B]] + \frac{\lambda^3}{3!}[A, A, [A, B]]] + \ldots && (36)
\end{aligned}$$

(b) Let $A(t)$ be an operator that depends on time. Derive the following operator identity:

$$\frac{d\left(e^{A}\right)}{dt} e^{-A} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} L_A^n \left(\frac{dA}{dt}\right), \tag{37}$$

where

$$L_A(X) = [A, X], \quad L_A^2(X) = [A, [A, X]], \quad \ldots, \tag{38}$$

where X is an arbitrary operator. Also, $L_A^0 = X$. Do not assume that A commutes with dA/dt.

We can again derive a differential equation. Define

$$F(\alpha,t) = \frac{d\left(e^{\alpha A(t)}\right)}{dt} e^{-\alpha A(t)} \tag{39}$$

Then

$$\begin{aligned}
\frac{\partial F}{\partial \alpha} &= \alpha \frac{d}{dt}\left(e^{\alpha A(t)} A(t)\right) e^{-\alpha A(t)} - \alpha \frac{d\left(e^{\alpha A(t)}\right)}{dt} A(t) e^{-\alpha A(t)} && (40)\\
&= \alpha \frac{d\left(e^{\alpha A}\right)}{dt} A e^{-\alpha A} + \alpha e^{\alpha A}\frac{dA}{dt} e^{-\alpha A} - \alpha \frac{d\left(e^{\alpha A}\right)}{dt} A e^{-\alpha A} && (41)\\
&= \alpha e^{\alpha A}\frac{dA}{dt} e^{-\alpha A} && (42)
\end{aligned}$$

Using the previous result with $B = dA/dt$, we can write

$$\begin{aligned}
\frac{\partial F}{\partial \alpha} &= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \underbrace{[A,[A,[A,\ldots,}_{n \text{ times}} \frac{dA}{dt}]]\ldots] && (43)\\
&= \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} L_A^n\left(\frac{dA}{dt}\right) && (44)
\end{aligned}$$

If we note that $F(0,t) = 0$, we can write

$$\begin{aligned}
F(\alpha,t) &= \int_0^{\alpha} d\alpha' \frac{\partial F}{\partial \alpha'}(\alpha',t) && (45)\\
&= \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{(n+1)!} L_A^n\left(\frac{dA}{dt}\right) && (46)
\end{aligned}$$

We can set $\alpha = 1$ to obtain the final result

$$F(1,t) = \frac{d\left(e^{A}\right)}{dt} e^{-A} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} L_A^n\left(\frac{dA}{dt}\right). \tag{47}$$

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3. (a) If A and B are square matrices, show that

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \tag{48}$$

Show that this implies the cyclic property of the trace of an arbitrary number of matrices

$$\mathrm{tr}(A_1A_2\dots A_n) = \mathrm{tr}(A_nA_1A_2\dots A_{n-1}) = \mathrm{tr}(A_{n-1}A_nA_1A_2\dots A_{n-2}) = \dots \tag{49}$$

In index notation

$$\mathrm{tr}(AB) = \sum_i (AB)_{ii} \tag{50}$$

$$= \sum_{ij} A_{ij}B_{ji} \tag{51}$$

Each component of the matrix is an ordinary number that commutes, so

$$\mathrm{tr}(AB) = \sum_{ij} B_{ji}A_{ij} \tag{52}$$

$$= \sum_j (BA)_{jj} \tag{53}$$

$$= \mathrm{tr}(BA) \tag{54}$$

This works for any number of matrices. E.g.,

$$\mathrm{tr}(ABCD) = \sum_{ijk\ell} A_{ij}B_{jk}C_{k\ell}D_{\ell i} \tag{55}$$

$$= \sum_{ijk\ell} D_{\ell i}A_{ij}B_{jk}C_{k\ell} = \mathrm{tr}(DABC) \tag{56}$$

$$= \sum_{ijk\ell} C_{k\ell}D_{\ell i}A_{ij}B_{jk} = \mathrm{tr}(CDAB) \tag{57}$$

$$= \sum_{ijk\ell} B_{jk}C_{k\ell}D_{\ell i}A_{ij} = \mathrm{tr}(BCDA) \tag{58}$$

but only cyclic permutations. E.g.,

$$\mathrm{tr}(ABCD) \neq \mathrm{tr}(BACD), \tag{59}$$

etc.

(b) Show that $\mathrm{tr}\sigma_i = 0$. Then use Eq. (2) to find a simple expression for $\mathrm{tr}(\sigma_i\sigma_j)$

By inspection, the sum of the diagonal elements are zero,

$$\mathrm{tr}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathrm{tr}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \mathrm{tr}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 0 \tag{60}$$

From Eq. (2), we have

$$\mathrm{tr}(\sigma_i\sigma_j) = \mathrm{tr}\left(\mathbb{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k\right) \tag{61}$$

$$= 2\delta_{ij}, \tag{62}$$

since $\mathrm{tr}(\mathbb{1}) = 2$.

(c) The set of four 2×2 matrices, $(\mathbb{1}, \boldsymbol{\sigma})$ form a basis in the space of 2×2 matrices. I.e., an arbitrary matrix M can be expressed as a linear combination of these four matrices:

$$M = a\mathbb{1} + \boldsymbol{b}\cdot\boldsymbol{\sigma}. \tag{63}$$

Find simple expressions for the expansion coefficients, a and $\boldsymbol{b} = (b_1, b_2, b_3)$. Do this by taking traces, or by multiplying by a Pauli matrix and then taking traces. Also show that M is Hermitian if and only if $a, \boldsymbol{b}$ are real.

$$M = a\mathbb{1} + \boldsymbol{b}\cdot\boldsymbol{\sigma} \tag{64}$$

$$\mathrm{tr}(M) = 2a \tag{65}$$

$$\implies a = \frac{1}{2}\mathrm{tr}(M) \tag{66}$$

$$\mathrm{tr}(\sigma_i M) = \mathrm{tr}\left(a\sigma_i + b_j\sigma_i\sigma_j\right) \tag{67}$$

$$= 0 + 2b_j\delta_{ij} \tag{68}$$

$$\implies b_i = \frac{1}{2}\mathrm{tr}(\sigma_i M) \tag{69}$$

Note that all the Pauli matrices are Hermitian: $\sigma_i^\dagger = \sigma_i$, and also the identity matrix. So the Hermitian conjugate of M is

$$M^\dagger = a^*\mathbb{1} + \boldsymbol{b}^*\cdot\boldsymbol{\sigma} \tag{70}$$

and

$$a^* = \frac{1}{2}\text{tr}(M^\dagger) \tag{71}$$
$$b_i^* = \frac{1}{2}\text{tr}(\sigma_i M^\dagger) \tag{72}$$

If $M = M^\dagger$, then $a = a^*$ and $b = b^*$, and also the converse.

(d) Now suppose that M is nonzero in only one of the 4 components, where it has value 1. That is, let

$$M_{ij} = \delta_{ir}\delta_{js} \tag{73}$$

For some coordinate (r,s).

Use this in Eq. (63) to find a nice expression for $(\sigma_m)_{ij}(\sigma_m)_{kl}$ (summing over all indices) in terms of Kronecker deltas.

We again write

$$M = a\mathbb{1} + \boldsymbol{b}\cdot\boldsymbol{\sigma} \tag{74}$$

with

$$a = \frac{1}{2}\text{tr}(M) \tag{75}$$
$$= \frac{1}{2}\sum_i M_{ii} \tag{76}$$
$$= \frac{1}{2}\sum_i \delta_{ir}\delta_{is} \tag{77}$$
$$= \frac{1}{2}\delta_{rs} \tag{78}$$
$$b_i = \frac{1}{2}\text{tr}(\sigma_i M) \tag{79}$$
$$= \frac{1}{2}\sum_{jk}(\sigma_i)_{jk}M_{kj} \tag{80}$$
$$= \frac{1}{2}\sum_{jk}(\sigma_i)_{jk}\delta_{kr}\delta_{js} \tag{81}$$
$$= \frac{1}{2}(\sigma_i)_{sr} \tag{82}$$

So

$$M = \frac{1}{2}\delta_{rs}\mathbb{1} + \frac{1}{2}\sum_m (\sigma_m)_{sr}\sigma_m \tag{83}$$

$$M_{ij} = \delta_{ir}\delta_{js} = \frac{1}{2}\delta_{rs}\delta_{ij} + \frac{1}{2}(\sigma_m)_{sr}(\sigma_m)_{ij} \tag{84}$$

and finally

$$(\sigma_m)_{sr}(\sigma_m)_{ij} = 2\delta_{ir}\delta_{js} - \delta_{rs}\delta_{ij} \tag{85}$$

or, changing indices,

$$(\sigma_m)_{ij}(\sigma_m)_{k\ell} = 2\delta_{kj}\delta_{\ell i} - \delta_{ij}\delta_{kl}. \tag{86}$$
