Solutions:
Homework Set 12

Due November 30, 2020

1. Consider a one-dimensional simple harmonic oscillator with classical angular frequency ω_0.

   It is in the ground state until time $t = 0$, when a small time-dependent potential is turned on

   $$V(t) = F_0 \; x \cos(\omega t), \tag{1}$$

   where F_0 is a constant.

   Obtain the expectation $\langle x \rangle$ as a function of time to first order in time-dependent perturbation theory.

   Is this procedure valid for $\omega \simeq \omega_0$?

   Note the useful relation for energy eigenstates of the simple harmonic oscillator

   $$\langle n' | \hat{x} | n \rangle = \sqrt{\frac{\hbar}{2m\omega_0}} \left(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1} \right) \tag{2}$$

In the interaction picture, the expectation value is

$$\begin{aligned} \langle x \rangle &= \langle \psi_S(t) | x | \psi_S(t) \rangle & (3) \\ &= \langle \psi_I(t) | x_I | \psi_I(t) \rangle & (4) \end{aligned}$$

with

$$x_I = e^{iH_0 t/\hbar} x e^{-iH_0 t/\hbar} \tag{5}$$

and

$$|\psi_I(t)\rangle = \sum_n c_n(t)|n\rangle \tag{6}$$

$$\implies \langle x\rangle = \sum_{m,n} e^{-i(E_n-E_m)t/\hbar} c_n(t)c_m^*(t)\langle m|x|n\rangle \tag{7}$$

To first order in perturbation theory, the transition amplitude for a state that is initially in the ground state $|0\rangle$ at time $t = 0$ is

$$c_0(t) = 1 \tag{8}$$

$$c_n(t) \simeq c_n^{(1)}(t) = \frac{1}{i\hbar}\int_0^t dt' e^{i(E_n-E_0)t'/\hbar}\langle n|H_1(t')|0\rangle, \tag{9}$$

$$= \frac{F_0}{i\hbar}\langle n|x|0\rangle \int_0^t dt' e^{i\omega_0 n t'}\cos(\omega t') \tag{10}$$

$$= \frac{F_0}{i\hbar}\sqrt{\frac{\hbar n}{2m\omega_0}}\delta_{n,1}\int_0^t dt' e^{i\omega_0 n t'}\cos(\omega t') \tag{11}$$

So the only non-zero contributions are

$$c_1(t) \simeq c_1^{(1)}(t) = \frac{F_0}{i\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\int_0^t dt' e^{i\omega_0 t'}\cos(\omega t') \tag{12}$$

$$= \frac{F_0}{2i}\sqrt{\frac{1}{2m\omega_0\hbar}}\int_0^t dt' \left(e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}\right) \tag{13}$$

$$= -\sqrt{\frac{F_0^2}{8m\omega_0\hbar}}\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right] \tag{14}$$

and

$$c_0(t) \simeq 1 \tag{15}$$

Finally, we have

$$\langle x \rangle = \sum_{m,n} e^{-i(E_n - E_m)t/\hbar} c_n(t) c_m^*(t) \langle m|x|n\rangle \tag{16}$$

$$= e^{-i(E_0 - E_1)t/\hbar} c_0(t) c_1^*(t) \langle 1|x|0\rangle + e^{-i(E_1 - E_0)t/\hbar} c_1(t) c_0^*(t) \langle 0|x|1\rangle \tag{17}$$

$$= -e^{i\omega_0 t} \sqrt{\frac{F_0^2}{8m\omega_0\hbar}} \left[\frac{e^{-i(\omega_0+\omega)t} - 1}{\omega_0 + \omega} + \frac{e^{-i(\omega_0-\omega)t} - 1}{\omega_0 - \omega} \right] \sqrt{\frac{\hbar}{2m\omega_0}} + \text{ complex conjugate} \tag{18}$$

$$= -\frac{F_0}{4m\omega_0} \left[\frac{e^{-i\omega t} - e^{i\omega_0 t}}{\omega_0 + \omega} + \frac{e^{i\omega t} - e^{i\omega_0 t}}{\omega_0 - \omega} \right] + \text{ c.c.} \tag{19}$$

$$= -\frac{F_0}{2m\omega_0} \left[\frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega_0 + \omega} + \frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega_0 - \omega} \right] \tag{20}$$

$$= \frac{F_0}{m} \frac{\cos(\omega t) - \cos(\omega_0 t)}{\omega^2 - \omega_0^2} \tag{21}$$

For $\omega \simeq \omega_0$,

$$\langle x \rangle \simeq -\frac{F_0}{2m} \frac{t \sin(\omega_0 t)}{\omega_0}, \tag{22}$$

and the perturbative corrections are no longer small except at very small times $t \ll 1/\omega \sim 1/\omega_0$. At larger times, high order corrections can be larger than lower order corrections, and the perturbation expansion breaks down.