Homework Set 1

Due August 24, 2020

1. Consider the 2x2 matrices:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

The three matrices σ_x, σ_y, and σ_z are called the *Pauli matrices*, which can also be denoted $\sigma_1, \sigma_2, \sigma_3$, respectively

(a) Show that

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \tag{2}$$

with $i \in \{1, 2, 3\}$. Here δ_{ij} is understood to be multiplied by $\mathbb{1}$, we use the summation convention (repeated indices k are summed) and ϵ_{ijk} is the completely antisymmetric Levi-Civita symbol.

(b) Consider a vector operator — that is, a set of operators $\boldsymbol{A} = (A_x, A_y, A_z)$, where A_x, A_y, A_x are operators.

Given two vector operators $\boldsymbol{A}, \boldsymbol{B}$ that commute with $\boldsymbol{\sigma}$ but not each other, use Eq. (2) to show that

$$(\boldsymbol{\sigma} \cdot \boldsymbol{A})(\boldsymbol{\sigma} \cdot \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{B} + i\boldsymbol{\sigma} \cdot (\boldsymbol{A} \times \boldsymbol{B}), \tag{3}$$

where

$$\boldsymbol{\sigma} = \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}, \tag{4}$$

and two vector operators commute if all their components commute.

Note that in general, $\boldsymbol{A} \cdot \boldsymbol{B} \neq \boldsymbol{B} \cdot \boldsymbol{A}$, and $\boldsymbol{A} \times \boldsymbol{B} \neq -\boldsymbol{B} \times \boldsymbol{A}$

(c) Let $\hat{n}$ be an arbitrary unit vector and θ an arbitrary angle. Show that

$$\exp(-i\theta\boldsymbol{\sigma}\cdot\hat{n}) = \mathbb{1}\cos\theta - i(\boldsymbol{\sigma}\cdot\hat{n})\sin\theta \tag{5}$$

2. (a) Consider two operators A, B that do not commute. Show that

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots \tag{6}$$

Hint: Replace A by λA, where λ is a parameter, and let the left-hand side be $F(\lambda)$. Find a differential equation satisfied by $F(\lambda)$ and solve it. Alternatively, expand $F(\lambda)$ in a Taylor series in λ. At the end, set $\lambda = 1$.

(b) Let $A(t)$ be an operator that depends on time. Derive the following operator identity:

$$\frac{d\left(e^A\right)}{dt}e^{-A} = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}L_A^n\left(\frac{dA}{dt}\right), \tag{7}$$

where

$$L_A(X) = [A,X], \quad L_A^2(X) = [A,[A,X]], \quad \ldots, \tag{8}$$

where X is an arbitrary operator. Also, $L_A^0 = X$. Do not assume that A commutes with dA/dt.

3. (a) If A and B are square matrices, show that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \tag{9}$$

Show that this implies the cyclic property of the trace of an arbitrary number of matrices

$$\mathrm{tr}(A_1A_2\ldots A_n) = \mathrm{tr}(A_nA_1A_2\ldots A_{n-1}) = \mathrm{tr}(A_{n-1}A_nA_1A_2\ldots A_{n-2}) = \ldots \tag{10}$$

(b) Show that $\mathrm{tr}\sigma_i = 0$. Then use Eq. (2) to find a simple expression for $\mathrm{tr}(\sigma_i\sigma_j)$

(c) The set of four 2×2 matrices, $(\mathbb{1}, \boldsymbol{\sigma})$ form a basis in the space of 2×2 matrices. I.e., an arbitrary matrix M can be expressed as a linear combination of these four matrices:

$$M = a\mathbb{1} + \boldsymbol{b} \cdot \boldsymbol{\sigma}. \tag{11}$$

Find simple expressions for the expansion coefficients, a and $\boldsymbol{b} = (b_1, b_2, b_3)$. Do this by taking traces, or by multiplying by a Pauli matrix and then taking traces. Also chow that M is Hermitian if and only if $a, \boldsymbol{b}$ are real.

(d) Now suppose that M is nonzero in only one of the 4 components, where it has value 1. That is, let

$$M_{ij} = \delta_{ir}\delta_{js} \tag{12}$$

For some coordinate (r,s).

Use this in Eq. (11) to find a nice expression for $(\sigma_m)_{ij}(\sigma_m)_{kl}$ (summing over all indices) in terms of Kronecker deltas.