Homework Set 9

Due October 23

1. Consider the wave function of a particle with spin (not necessarily $s = 1/2$) moving in 3 dimensions.

$$\psi(\mathbf{x}, m) = \langle \mathbf{x}, m|\psi\rangle = \psi_m(\mathbf{x}) \tag{1}$$

   (a) If $\psi_m(\mathbf{r})$ is the wave function of a particle in state $|\psi\rangle$, then what is the wave function of the particle in the rotated state $U(R)|\psi\rangle$? The rotation operator $U(R)$ is the product of a spatial rotation times a spin rotation, both parameterized by the same R.

   (b) Consider the specific case of a spin-1/2 electron moving in an electromagnetic field, plus a central force potential $V(r)$. The full Hamiltonian is

$$H = \frac{1}{2m}\left[\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{x}, \mathbf{t})\right]^2 + q\Phi(\mathbf{x}, t) - \mu \cdot \mathbf{B}(\mathbf{x}, t) + V(r). \tag{2}$$

   For an electron we have $q = -e$ and $g = g_e \simeq 2$, so that $\mu = -g_e\mu_B\mathbf{S}/\hbar$.

   Let the magnetic field be uniform, $\mathbf{B} = B\hat{b}$, and choose a gauge such that $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{x}$ and $\Phi = 0$. Let $\omega_0 = eB/mc$.

   Consider the time-dependent Schrödinger equation for the electron,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{3}$$

   Define a new state $|\phi(t)\rangle$ by

$$|\psi(t)\rangle = U(\hat{b}, \omega t)|\phi(t)\rangle, \tag{4}$$

   where $U(\hat{b}, \omega t)$ is a rotation operator that rotates the whole system (orbital and spin degrees of freedom). This means that $|\phi(t)\rangle$ is the state in a frame rotating with angular velocity ω about the axis $\hat{b}$.

   Find a frequency ω that eliminates the effect of the magnetic field on the orbital motion of the particle, apart from the centrifugal potential which is proportional to $(\mathbf{b} \times \mathbf{x})^2$.

Find a frequency ω that eliminates the effect of the magnetic field on the spin.

Express your answers as some multiple of ω_0. Can you eliminate the effects of the magnetic field entirely, apart from the centrifugal potential?

2. Add angular momenta $j_1 = 1$ and $j_2 = 1$ (e.g., two spin-1 particles, or a spin-1 particle in a $l = 1$ orbital state).

   What are the possible values for total angular momentum j?

   Express all possible $|j, m\rangle$ eigenkets in terms of tensor product eigenkets $|j_1 m_1; j_2 m_2\rangle$. Write your answer as

$$|j = 1, m = 1\rangle = \#|+, 0\rangle + \#|0, +\rangle, \ldots, \tag{5}$$

   where $+$, 0, and $-$ stand for $m_{1,2} = 1, 0, -1$ respectively.