Solutions:
Homework Set 7

Due October 9

1. Show that the 3×3 matrices $G_i (i = 1, 2, 3)$ whose elements are given by

$$(G_i)_{jk} = -i\hbar\epsilon_{ijk}, \tag{1}$$

where j and k are the row and column indices, satisfy the angular-momentum commutation relations.

What is the physical (or geometric) significance of the transformation matrix that connects G_i to the more usual 3×3 representations of the angular-momentum operator J_i with J_3 taken to be diagonal?

Relate your result to

$$\mathbf{V} \to \mathbf{V} + \hat{\mathbf{n}}\delta\phi \times \mathbf{V} \tag{2}$$

under infinitesimal rotations. (*Note*: This problem may be helpful in understanding the photon spin.)

It is useful to note the following identity of the Levi-Civita symbol

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} && (3) \\ &= \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) && (4) \end{aligned}$$

of which a special case is

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{5}$$

So the commutator is

$$
\begin{aligned}
[G_i, G_j]_{ln} &= (G_iG_j - G_jG_i)_{lm} = (G_i)_{lm}(G_j)_{mn} - (G_j)_{lm}(G_i)_{mn} && (6)\\
&= -\hbar^2 \left[\epsilon_{ilm}\epsilon_{jmn} - \epsilon_{jlm}\epsilon_{imn}\right] && (7)\\
&= -\hbar^2 \left[(\delta_{in}\delta_{lj} - \delta_{ij}\delta_{ln}) - (\delta_{jn}\delta_{li} - \delta_{ji}\delta_{ln})\right] && (8)\\
&= \hbar^2 \left(\delta_{jn}\delta_{li} - \delta_{in}\delta_{lj}\right) && (9)\\
&= \hbar^2\epsilon_{kij}\epsilon_{kln} && (10)\\
&= i\hbar\epsilon_{ijk}(G_k)_{ln} && (11)
\end{aligned}
$$

Which is the usual commutation relation for angular momentum

$$[G_i, G_j] = i\hbar\epsilon_{ijk}G_k. \tag{12}$$

The usual representation of J_i has J_3 diagonal. We can obtain this representation by finding the unitary transformation that diagonalizes G_3:

$$G_3 = i\hbar \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

The eigenvalues and eigenvectors are

$$\lambda = +\hbar, \quad \vec{r}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \tag{14}$$

$$\lambda = 0, \quad \vec{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{15}$$

$$\lambda = -\hbar, \quad \vec{r}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ i \\ 0 \end{pmatrix} \tag{16}$$

The matrix U that diagonalizes G_3 has each eigenvector as a column:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{17}$$

$$U^\dagger G_3 U = J_3 = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{18}$$

and likewise

$$J_1 = U^\dagger G_1 U \tag{19}$$

$$J_2 = U^\dagger G_2 U \tag{20}$$

Since the G_i and J_i both satisfy the angular momentum commutation relation, they both are representations of the rotation group. They are related to each other by a rotation U.

This finite rotation can be obtained by compounding infinitesimal rotations $\delta\phi$ around axis $\hat{n}$.

$$\mathbf{V} \to \mathbf{V}' = \mathbf{V} + \hat{\mathbf{n}}\delta\phi \times \mathbf{V} \tag{21}$$

$$\implies V_i' = [\delta_{ik} + \delta\phi\epsilon_{ijk}n_j]\, V_k \tag{22}$$

$$= [\delta_{ik} - \delta\phi\epsilon_{ijk}n_i]\, V_k \tag{23}$$

$$= \left[\delta_{ik} - \delta\phi\frac{i}{\hbar}(G_i)_{jk}n_i\right] V_k \tag{24}$$

$$\implies \mathbf{V}' = \left[\mathbb{1} - \frac{i}{\hbar}\delta\phi\hat{n}\cdot\mathbf{G}\right]\mathbf{V} \tag{25}$$

So for a finite rotation:

$$\mathbf{V}' = e^{-\frac{i}{\hbar}\delta\phi\hat{n}\cdot\mathbf{G}}\mathbf{V} \tag{26}$$

2. (a) Let **J** be an angular momentum operator. Using the fact that J_x, J_y, J_z satisfy the usual angular momentum commutation relations, and

$$J_\pm \equiv J_x \pm iJ_y, \tag{27}$$

prove

$$J^2 = J_z^2 + J_+J_- - \hbar J_z. \tag{28}$$

$$\begin{aligned}
J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) && (29)\\
&= J_x^2 + J_y^2 - iJ_yJ_x - iJ_xJ_y && (30)\\
&= J_x^2 + J_y^2 - i[J_x, J_y] && (31)\\
&= J^2 - J_z^2 + \hbar J_z && (32)\\
\implies J^2 &= J_z^2 + J_+J_- - \hbar J_z && (33)
\end{aligned}$$

(b) Using this, derive the coefficient c_- defined by

$$J_-|jm\rangle = c_-|j, m-1\rangle \tag{34}$$

For $m > -j$ the magnitude is given by

$$\begin{aligned}
\langle jm|J_+J_-|jm\rangle = |c_-^-|^2 &= \langle jm|(J^2 - J_3^2 + \hbar J_3)|jm\rangle && (35)\\
&= (j+m)(j-m+1) && (36)
\end{aligned}$$

The phase is arbitrary, until a convention is chosen for the phases of the states $|jm\rangle$. Normally, these are chosen so that c_- is real and positive:

$$c_- = \sqrt{(j+m)(j-m+1)} \tag{37}$$

3. Consider an orbital angular-momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in m=0, ±1, and ±2.

The probability is

$$P(m) = \left|\langle 2, m|e^{-i\hbar\beta J_y/\hbar}|2,0\rangle\right|^2 \tag{38}$$
$$= |d^2_{m,0}|^2 \tag{39}$$

We can look up the form of $d^2_{m,m'}$ in a reference, or we can use Wigner's formula.

$$d^j_{m',m} = \sum_k (-1)^{k-m+m'} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(k-m+m')!} \tag{40}$$
$$\times \left(\cos\frac{\beta}{2}\right)^{2j-2k+m-m'} \left(\sin\frac{\beta}{2}\right)^{2k-m+m'} \tag{41}$$
$$\implies d^2_{m,0} = \sum_k (-1)^{k-m} \frac{\sqrt{4(2+m)!(2-m)!}}{(2-k)!k!(2-k-m)!(k+m)!} \left(\cos\frac{\beta}{2}\right)^{4-2k-m} \left(\sin\frac{\beta}{2}\right)^{2k+m}, \tag{42}$$

where terms in the sum are included only when the factorials are not negative.

Note that $d^j_{m',m} = d^j_{-m',-m}$, and so the probabilities with positive and negative m are the same.

So for $m = 2$, we have only the $k = 0$ term

$$d^2_{2,0} = d^2_{-2,0} = (-1)^{-2} \frac{\sqrt{4(2+2)!(2-2)!}}{(2)!0!(2-2)!(0+2)!} \left(\cos\frac{\beta}{2}\right)^{4-2} \left(\sin\frac{\beta}{2}\right)^{2} \tag{43}$$
$$= \frac{\sqrt{4*4!}}{2^2}\frac{1}{4}\sin^2\beta \tag{44}$$
$$= \frac{\sqrt{6}}{4}\sin^2\beta \tag{45}$$

For $m = 1$, we have terms with $k = 0$ and $k = 1$.

$$d^2_{1,0} = -\frac{\sqrt{4*3!*1!}}{2!}\cos^3\frac{\beta}{2}\sin\frac{\beta}{2} + \frac{\sqrt{24}}{2}\cos\frac{\beta}{2}\sin^3\frac{\beta}{2} \tag{46}$$

$$= \frac{\sqrt{6}}{2}\sin\beta(\sin^2\frac{\beta}{2} - \cos^2\frac{\beta}{2}) \tag{47}$$

$$= \frac{\sqrt{6}}{2}\sin\beta\cos\beta \tag{48}$$

$$= \frac{\sqrt{6}}{4}\sin 2\beta \tag{49}$$

For $m = 0$, we have all 3 terms with $k = 0, 1, 2$.

$$d^2_{0,0} = \frac{\sqrt{4*2!*2!}}{2!*2!}\cos^4\frac{\beta}{2} - \frac{\sqrt{16}}{1}\cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} + \frac{4}{2!2!}\sin^4\frac{\beta}{2} \tag{50}$$

$$= \cos^4\frac{\beta}{2} + \sin^4\frac{\beta}{2} - 4\cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2} \tag{51}$$

$$= \frac{1}{4}(1 + 3\cos^2 2\beta) \tag{52}$$

So the probability of each of the 5 measurements is

$$P(2) = P(-2) = |d^2_{\pm 2,0}(\beta)|^2 = \frac{3}{8}\sin^4\beta \tag{53}$$

$$P(1) = P(-1) = |d^2_{\pm 1,0}(\beta)|^2 = \frac{3}{8}\sin^2(2\beta) \tag{54}$$

$$P(0) = |d^2_{0,0}(\beta)|^2 = \frac{1}{16}(3\cos^2 2\beta + 1)^2 \tag{55}$$

4. A spin-1 particle has the component of its spin in the direction

$$\hat{n} = \frac{1}{\sqrt{3}}(1, 1, 1), \tag{56}$$

measured, and the result is $\hbar$. Subsequently S_z is measured, with various probabilities of the three possible outcomes.

Let R be a rotation that maps the $\hat{z}$ axis into $\hat{n}$,

$$R\hat{z} = \hat{n}. \tag{57}$$

Express the probabilities of the three possible measurement outcomes in terms of the rotation matrix

$$D^1_{mm'}(R) \equiv \langle j = 1, m'|U(R)|j = 1, m\rangle \tag{58}$$

Work out this matrix to find the probabilities explicitly. (You may use tables of d-matrices).

13.2 We have a (spin-1) eigenstate of the spin as measured in the $\hat{n}$-direction. Call this state $|\hat{n};m\rangle$

By definition, $S_n|\hat{n};m\rangle = m|\hat{n};m\rangle$ $(\hbar = 1)$

& $S_n = \hat{n}\cdot\vec{S}$

The spin operators in this system, $\vec{S}$, are the angular momentum operators & so are subject to the adjoint identity:

$$\mathcal{U}(R)\,\vec{S}\,\mathcal{U}^\dagger(R) = \overleftrightarrow{R}^{-1}\vec{S}$$ (the arrow over the R indicates that it's only meant to act on vectors)

Let $|m\rangle$ be eigenstates of S_z: $S_z|m\rangle = m|m\rangle$

& R be a (non-unique) rotation such that $R(\hat{z}) = \hat{n}$

Claim: $|\hat{n};m\rangle = \mathcal{U}(R)|m\rangle$ (up to a phase)

pf: $S_n(\mathcal{U}(R)|m\rangle) = \hat{n}\cdot\vec{S}\,\mathcal{U}(R)|m\rangle$

$= \mathcal{U}(R)\,\mathcal{U}^\dagger(R)\,\hat{n}\cdot\vec{S}\,\mathcal{U}(R)|m\rangle$

$= \hat{n}\cdot\left(\mathcal{U}(R)\left[\mathcal{U}^\dagger(R)\vec{S}\,\mathcal{U}(R)\right]\right)|m\rangle$

$\mathcal{U}(R^{-1}) = [\mathcal{U}(R)]^{-1} = \mathcal{U}^\dagger(R) \Rightarrow$ $= \hat{n}\cdot\left(\mathcal{U}(R)\left[\mathcal{U}(R^{-1})\vec{S}\,\mathcal{U}^\dagger(R^{-1})\right]\right)|m\rangle$

$= \mathcal{U}(R)\,\hat{n}\cdot(\overleftrightarrow{R}\vec{S})\,|m\rangle$

$\vec{a}\cdot\vec{b} = R(\vec{a})\cdot R(\vec{b})$ $\Rightarrow$ $= \mathcal{U}(R)\left(R^{-1}(\hat{n})\cdot\vec{S}\right)|m\rangle$

$R(\hat{z}) = \hat{n} \Rightarrow R^{-1}(\hat{n}) = \hat{z}$ $\Rightarrow$ $= \mathcal{U}(R)(\hat{z}\cdot\vec{S})|m\rangle = \mathcal{U}(R)S_z|m\rangle = \mathcal{U}(R)m|m\rangle$

$= m(\mathcal{U}(R)|m\rangle)$

$\Rightarrow$ $\mathcal{U}(R)|m\rangle$ is an eigenket of S_n ✓

□

– Starting state = $|\hat{n};+1\rangle$ w/ $\hat{n} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

The probability for ~~[struck]~~ measuring m in a measurement of S_z, then,

is:

$$P_m = |\langle m|\hat{n};+1\rangle|^2$$

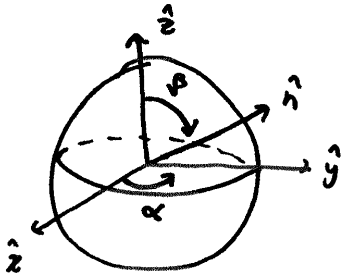
$$\langle m|\hat{n};+1\rangle = \langle m|\mathcal{U}(R)|+1\rangle = D^{1}_{m1}(R) \quad (j=1)$$

Side Note: Let R be expressed in Euler angles α, β, γ

$$\Rightarrow \mathcal{U}(R) = \mathcal{U}_z(\alpha)\,\mathcal{U}_y(\beta)\,\mathcal{U}_z(\gamma)$$

Then,
$$D^j_{mm'}(\alpha,\beta,\gamma) = \langle j\, m|\mathcal{U}(R)|j\, m'\rangle$$
$$= \langle j\, m|\mathcal{U}_z(\alpha)\,\mathcal{U}_y(\beta)\,\mathcal{U}_z(\gamma)|j\, m'\rangle$$
$$= e^{im\alpha}\langle j\, m|\mathcal{U}_y(\beta)|j\, m'\rangle e^{-im'\gamma}$$
$$= e^{i(m\alpha - m'\gamma)} d^j_{mm'}(\beta) \qquad \left(d^j_{mm'}(\beta) = [d^j_{m'm}(-\beta)]^*\right)$$

$$\Rightarrow P_m = |D^1_{m1}(R)|^2 = |e^{i(m\alpha - m'\gamma)} d^1_{m1}(\beta)|^2 = |d^1_{m1}(\beta)|^2 = |d^1_{1m}(-\beta)|^2$$



$$\Rightarrow \hat{z}\cdot\hat{n} = \cos\beta$$

$$\Rightarrow P_m = |d^1_{1m}(-\cos^{-1}(\hat{z}\cdot\hat{n}))|^2$$

$$d^1_{1\pm1}(\beta) = \frac{1\pm\cos\beta}{2} \Rightarrow P_{\pm1} = \left|\frac{1\pm(\hat{z}\cdot\hat{n})}{2}\right|^2 \qquad \hat{n} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow \hat{z}\cdot\hat{n} = \frac{1}{\sqrt{3}}$$

$$d^1_{10}(\beta) = \frac{-\sin\beta}{\sqrt{2}} \Rightarrow P_0 = \left|\frac{\sqrt{1-(\hat{z}\cdot\hat{n})^2}}{\sqrt{2}}\right|^2$$

$$\Rightarrow \boxed{P_{\pm1} = \frac{1}{3} \pm \frac{\sqrt{3}}{6} \qquad P_0 = \frac{1}{3}} \qquad (P_1 + P_0 + P_{-1}) = 1 \checkmark$$