

Solutions:

Homework Set 6

Due October 2

1. Find the eigenvalues and eigenvectors of

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (1)$$

Suppose an electron is in the spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. If S_y is measured, what is the probability of the result $+\hbar/2$, in terms of α and β ?

The two eigenvalues are $\lambda_{\pm} = \pm 1$. The corresponding eigenvectors are

$$\psi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad (2)$$

$$\psi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad (3)$$

The spin operator is $S_y = \hbar/2\sigma_y$, and so has eigenvalues $\pm\hbar/2$.

The probability of measuring $+\hbar/2$ is

$$\frac{1}{\alpha^2 + \beta^2} \left| \langle \psi_+ | \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rangle \right|^2 = \frac{1}{\alpha^2 + \beta^2} \left| \frac{1}{\sqrt{2}} (i\alpha + \beta) \right|^2 \quad (4)$$

$$= \frac{\alpha^2 + \beta^2 + 2\text{Im}(\alpha\beta^*)}{2(\alpha^2 + \beta^2)} \quad (5)$$

$$= \frac{1}{2} - \frac{\text{Im}(\alpha\beta^*)}{\alpha^2 + \beta^2} \quad (6)$$

2. In the 2-dimensional space of a spin-1/2 particle, consider a sequence of Euler rotations

represented by

$$U(\alpha, \beta, \gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right) \exp\left(\frac{-i\sigma_2\beta}{2}\right) \exp\left(\frac{-i\sigma_3\gamma}{2}\right) \quad (7)$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2} \cos \frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2} \sin \frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2} \sin \frac{\beta}{2} & e^{i(\alpha+\gamma)/2} \cos \frac{\beta}{2} \end{pmatrix}. \quad (8)$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ . Find θ .

In terms of an axis \hat{n} and an angle θ , the rotation operator in spin-1/2 space is (see, e.g., Sakurai 3.2.45)

$$U(\hat{n}, \theta) = e^{-\frac{i\theta}{2}\hat{n}\cdot\vec{\sigma}} = \begin{pmatrix} \cos \frac{\theta}{2} - in_z \sin \frac{\theta}{2} & (-in_x - n_y) \sin \frac{\theta}{2} \\ (-in_x + n_y) \sin \frac{\theta}{2} & \cos \frac{\theta}{2} + in_z \sin \frac{\theta}{2} \end{pmatrix} \quad (9)$$

Setting these equal, one can solve for (\hat{n}, θ) in terms of (α, β, γ) , or vice versa. The easiest way to solve for θ is to equate the trace of each matrix

$$\text{Tr}U(\hat{n}, \theta) = 2 \cos \frac{\theta}{2} \quad (10)$$

$$= \text{Tr}U(\alpha, \beta, \gamma) = 2 \cos \frac{\beta}{2} \cos \frac{\alpha + \gamma}{2} \quad (11)$$

$$\implies \theta = 2 \cos^{-1} \left[2 \cos \frac{\beta}{2} \cos \frac{\alpha + \gamma}{2} \right] \quad (12)$$

3. (a) Consider a pure ensemble of identically prepared spin-1/2 systems. Suppose the expectation values $\langle S_x \rangle$ and $\langle S_z \rangle$ are known, as well as the sign of $\langle S_y \rangle$.

Show how we may determine the state vector. Why is it unnecessary to know the magnitude of $\langle S_y \rangle$?

A general state in a spin 1/2 system can be written as a superposition of eigenstates of S_z

$$|\psi\rangle = C_+|+\rangle + C_-|-\rangle \quad (13)$$

with

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle \quad (14)$$

$$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle \quad (15)$$

Normalization gives the condition $|C_+|^2 + |C_-|^2 = 1$, and so the state can be written as

$$|\psi\rangle = \cos \frac{\beta}{2} e^{i\phi_+} |+\rangle + \sin \frac{\beta}{2} e^{i\phi_-} |-\rangle \quad (16)$$

for some (real) parameter β . Since the overall phase of a ket is irrelevant, we can multiply by $e^{-i(\phi_+ + \phi_-)/2}$

$$|\psi\rangle = \cos \frac{\beta}{2} e^{i(\phi_+ - \phi_-)/2} |+\rangle + \sin \frac{\beta}{2} e^{-i(\phi_+ - \phi_-)/2} |-\rangle \quad (17)$$

$$= \cos \frac{\beta}{2} e^{i\alpha/2} |+\rangle + \sin \frac{\beta}{2} e^{-i\alpha/2} |-\rangle \quad (18)$$

with $\alpha = (\phi_+ - \phi_-)/2$.

So the state is completely determined by two (real) parameters α and β .

$\langle S_x \rangle$ and $\langle S_z \rangle$ are

$$\langle S_z \rangle = \frac{\hbar}{2} \langle \psi | [|+\rangle \langle +| - |-\rangle \langle -|] | \psi \rangle \quad (19)$$

$$= \frac{\hbar}{2} \left[\cos^2 \frac{\beta}{2} - \sin^2 \frac{\beta}{2} \right] \quad (20)$$

$$= \frac{\hbar}{2} \cos \beta \quad (21)$$

$$\langle S_x \rangle = \frac{\hbar}{2} \langle \psi | [|+\rangle \langle -| + |-\rangle \langle +|] | \psi \rangle \quad (22)$$

$$= \frac{\hbar}{2} \left[\cos \frac{\beta}{2} e^{-i\alpha/2} \sin \frac{\beta}{2} e^{-i\alpha/2} + \sin \frac{\beta}{2} e^{i\alpha/2} \cos \frac{\beta}{2} e^{i\alpha/2} \right] \quad (23)$$

$$= \hbar \sin \frac{\beta}{2} \cos \frac{\beta}{2} \cos \alpha \quad (24)$$

$$= \frac{\hbar}{2} \sin \beta \cos \alpha \quad (25)$$

so knowing these, we can solve for β and $\cos \alpha$.

For the average $\langle S_y \rangle$, we have

$$\langle S_y \rangle = -i \frac{\hbar}{2} \langle \psi | [|+ \rangle \langle - | - | - \rangle \langle + |] | \psi \rangle \quad (26)$$

$$= -i \frac{\hbar}{2} \left[\cos \frac{\beta}{2} e^{-i\alpha/2} \sin \frac{\beta}{2} e^{-i\alpha/2} - \sin \frac{\beta}{2} e^{i\alpha/2} \cos \frac{\beta}{2} e^{i\alpha/2} \right] \quad (27)$$

$$= -\frac{\hbar}{2} \sin \beta \sin \alpha. \quad (28)$$

If we already know β and $\cos \alpha$, then we only need go know the sign of $\sin \alpha$, and so we only need the sign of $\langle S_y \rangle$ to completely determine the system.

- (b) Consider a mixed ensemble of spin-1/2 systems. Suppose the ensemble averages $[S_x]$, $[S_y]$, and $[S_z]$ are all known. Show how we may construct the 2×2 density matrix that characterizes the ensemble.

Again using the S_z basis, we can parameterize the density matrix as

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (29)$$

The ensemble averages are

$$[S_x] = \text{Tr}(\rho S_x) = \frac{\hbar}{2} \text{Tr} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \frac{\hbar}{2} (b + c) \quad (30)$$

$$[S_y] = \frac{\hbar}{2} \text{Tr} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right] = i \frac{\hbar}{2} (b - c) \quad (31)$$

$$[S_z] = \frac{\hbar}{2} \text{Tr} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \frac{\hbar}{2} (a - c) \quad (32)$$

We also need the normalization condition for any density matrix

$$\text{Tr} \rho = (a + d) = 1 \quad (33)$$

Solving the 4 equations for (a,b,c,d):

$$a = \langle +|\rho|+\rangle = \frac{1}{\hbar} \left([S_z] + \frac{\hbar}{2} \right) \quad (34)$$

$$b = \langle +|\rho|-\rangle = \frac{1}{\hbar} ([S_x] - i[S_y]) \quad (35)$$

$$c = \langle -|\rho|+\rangle = \frac{1}{\hbar} ([S_x] + i[S_y]) \quad (36)$$

$$d = \langle -|\rho|-\rangle = -\frac{1}{\hbar} \left([S_z] - \frac{\hbar}{2} \right) \quad (37)$$