

# Solutions:

## Homework Set 5

Due September 18

1. As derived in class, the eigenfunctions  $\psi_n(x) = \langle x | \psi_n \rangle$  of the harmonic oscillator in configuration space are given by

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{n!2^n}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-m\omega x^2/2\hbar} \quad (1)$$

with  $H_n$  the Hermite polynomials, satisfying the Rodriguez formula

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}. \quad (2)$$

In this problem, you may use dimensionless units,  $m = \omega = \hbar = 1$ , as done in class.

Find the corresponding momentum space wave eigenfunctions,  $\psi(p) = \langle p | \psi \rangle$ .

The system has a symmetry with respect to  $x \leftrightarrow p$ , which is especially evident in dimensionless units:

$$H = \frac{1}{2} (\hat{p}^2 + \hat{x}^2), \quad (3)$$

$\implies$  Up to a phase  $\alpha$ , the wavefunctions are the same as the configuration space wavefunctions:

$$\psi_n(p) = e^{i\alpha_n} \frac{1}{\sqrt{n!2^n}} H_n(p) e^{-p^2/2} \quad (4)$$

The phases are defined in configuration space by Eq. (1), which we must relate to the momentum-space wavefunctions.

The ground state in configuration space (in units  $x \leftrightarrow p$ ) is

$$\psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2} \quad (5)$$

The momentum space ground state is then a Fourier transform:

$$\psi_0(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \frac{1}{\pi^{1/4}} e^{-x^2/2} = \frac{1}{\pi^{1/4}} e^{-p^2/2} \quad (6)$$

So the phase  $\alpha_0 = 0$ . To get the phase of the other states we note

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle, \quad (7)$$

$$a^\dagger = \frac{x - ip}{\sqrt{2}} = \frac{-i}{\sqrt{2}} \left( p - \frac{d}{dp} \right), \quad (8)$$

$$\implies \psi_n(p) = \frac{(-i)^n}{\pi^{1/4}} \frac{1}{\sqrt{n!2^n}} \left( p - \frac{d}{dp} \right)^n e^{-p^2/2} \quad (9)$$

$$(10)$$

Comparing to the derivation of the configuration space wavefunction, we can see that  $e^{i\alpha_n} = (-i)^n$ , and

$$\psi_n(p) = \frac{(-i)^n}{\sqrt{n!2^n}} H_n(p) e^{-p^2/2} \quad (11)$$

2. In classical mechanics, any two Lagrangians that differ by a total time derivative produce the same equations of motion. For example, in one dimension,  $L$  and  $L'$ , defined by

$$L' = L + \frac{df(x, t)}{dt} = L + \frac{\partial f}{\partial t} + \dot{x} \frac{\partial f}{\partial x} \quad (12)$$

give the same equations of motion. This is easily verified by using both  $L$  and  $L'$  in the Euler-Lagrange equations,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \quad (13)$$

- (a) Let  $L_0$  be the classical Lagrangian for a free particle

$$L_0(x, \dot{x}) = \frac{m}{2} \dot{x}^2, \quad (14)$$

and  $L_S$  be the classical Lagrangian for a particle in a uniform gravitational field (with

$x$ -axis pointing up),

$$L_s(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - mgx \quad (15)$$

According to the principle of equivalence, motion in an accelerated frame is physically indistinguishable from motion in a uniform gravitational field.

Consider a region of space free of gravitational fields, where the particle motion in an inertial frame with coordinate  $x$  is described by Lagrangian  $L_0(x, \dot{x})$ . Let  $y$  be the coordinate in a frame that is accelerated at a constant acceleration  $g$  in the  $+x$  direction. Assume that the origins of the inertial frame ( $x$ ) and accelerated frame ( $y$ ) coincide at  $t = 0$ .

Transform  $L_0(x, \dot{x})$  to the  $y$  coordinate, and show that the result is  $L_s(y, \dot{y})$  plus the exact time derivative of a function  $f(y, t)$ :

$$L_0(x, \dot{x}) = L_s(y, \dot{y}) + \frac{d}{dt}f(y, t) \quad (16)$$

Determine  $f(y, t)$ .

$$L_0(x, \dot{x}) = \frac{m}{2}\dot{x}^2 \quad (17)$$

$$x = y + \frac{1}{2}gt^2 \quad (18)$$

$$\dot{x} = \dot{y} + gt \quad (19)$$

$$\implies L_0(x, \dot{x}) = \frac{m}{2}(\dot{y} + gt)^2 \quad (20)$$

$$= \frac{m}{2}(\dot{y}^2 + 2\dot{y}gt + g^2t^2) \quad (21)$$

$$= \frac{m}{2}\dot{y}^2 - mgy + mgy + m\dot{y}gt + \frac{m}{2}g^2t^2 \quad (22)$$

$$= L_g(y, \dot{y}) + \frac{d}{dt}\left(m\dot{y}gt + \frac{m}{6}g^2t^3\right) \quad (23)$$

$$f(y, t) = m\dot{y}gt + \frac{m}{6}g^2t^3 \quad (24)$$

- (b) Let  $H_0$  and  $H_g$  be the quantum Hamiltonians for a free particle and a particle in a uniform gravitational field,

$$H_0 = \frac{p^2}{2m}, \quad H_g = \frac{p^2}{2m} + mgx, \quad (25)$$

and let  $U_0(t)$  and  $U_g(t)$  be corresponding time evolution operators,

$$U_0(t) = e^{-iH_0t/\hbar}, \quad U_g(t) = e^{-iH_g t/\hbar}. \quad (26)$$

The propagator for the free particle is

$$\langle x_1|U_0(t)|x_0\rangle = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[\frac{i}{\hbar} \frac{m(x_1 - x_0)^2}{2t}\right]. \quad (27)$$

Use the path integral to find the propagator of a particle in a uniform gravitational field,  $\langle x_1|U_g(t)|x_0\rangle$ .

Hint: you do not need the detailed, discretized version of the path integral; instead, just use the compact form

$$\langle x_1|U(t)|x_0\rangle = \int d[x(\tau)] \exp\left(\frac{i}{\hbar} \int_0^t L d\tau\right) \quad (28)$$

and follow the obvious rules of calculus in manipulating it.

$$\langle x_1|U_0(t)|x_0\rangle = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left[\frac{i}{\hbar} \frac{m(x_1 - x_0)^2}{2t}\right] \quad (29)$$

$$= \int d[x(\tau)] \exp\left(\frac{i}{\hbar} \int_0^t d\tau L_0(x, \dot{x})\right) \quad (30)$$

$$= \int d[y(\tau)] \exp\left(\frac{i}{\hbar} \int_0^t d\tau L_s(y, \dot{y}) + \frac{d}{dt} f(y, t)\right) \quad (31)$$

Since  $y(\tau) = x(\tau) + \frac{1}{2}g\tau^2$ ,

$$\implies d[y(\tau)] = d[x(\tau)].$$

The paths satisfy

$$y(0) = x(0) = x_0 \quad (32)$$

$$y(t) = x(t) - \frac{1}{2}gt^2 = y_1 = x_1 - \frac{1}{2}gt^2 \quad (33)$$

So the path integral is

$$\langle x_1|U_0(t)|x_0\rangle = \int d[y(\tau)] \exp\left(\frac{i}{\hbar} \int_0^t d\tau L_s(y, \dot{y}) + \frac{i}{\hbar} [f(y_1, t) - f(y_0, 0)]\right) \quad (34)$$

with

$$f(y_1, t) - f(y_0, 0) = mgy_1t + \frac{m}{6}g^2t^3. \quad (35)$$

This is independent of the path  $y(\tau)$ , and can be taken out of the path integral

$$\langle x_1|U_0(t)|x_0\rangle = e^{\frac{i}{\hbar}(mgy_1t + \frac{m}{6}g^2t^3)} \int d[y(\tau)] \exp\left(\frac{i}{\hbar} \int_0^t d\tau L_s(y, \dot{y})\right) \quad (36)$$

$$= e^{\frac{i}{\hbar}(mgy_1t + \frac{m}{6}g^2t^3)} \langle y_1|U_g(t)|y_0\rangle \quad (37)$$

So, finally,

$$\langle y_1|U_g(t)|y_0\rangle = e^{-\frac{i}{\hbar}(mgy_1t + \frac{m}{6}g^2t^3)} \langle x_1|U_0(t)|x_0\rangle \quad (38)$$

$$= e^{-\frac{i}{\hbar}(mgy_1t + \frac{m}{6}g^2t^3)} \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{i}{\hbar} \frac{m[y_1 + \frac{1}{2}gt^2 - y_0]^2}{2t}} \quad (39)$$

Or, taking  $y \rightarrow x$ ,

$$\langle x_1|U_g(t)|x_0\rangle = \sqrt{\frac{m}{2\pi i\hbar t}} \exp \frac{i}{\hbar} \left[ \frac{m(x_1 - x_0)^2}{2t} - \frac{m}{2}gt(x_1 + x_0) - \frac{mg^2t^3}{24} \right] \quad (40)$$