Solutions:
Homework Set 3

Due August 28

1. The projection postulate of quantum mechanics says that if a system is described by a pure state $|\psi\rangle$ (here assumed to be normalized), then after a measurement of the operator A producing eigenvalue a_n, the system is described by the (normalized) state

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}, \tag{1}$$

where P_n is the projector onto the n-th eigenspace of A.

I.e., for no degeneracy

$$P_n = |n\rangle\langle n|. \tag{2}$$

When there is a degeneracy of order M, we define an orthonormal basis within the eigenspace, labeled by j, and

$$P_n = \sum_{j_1}^{M} |nj\rangle\langle nj| \tag{3}$$

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$$P_n = \sum_{j_1}^{M} |nj\rangle\langle nj| \tag{4}$$

Suppose instead the system is a mixed state described by a density operator ρ (assumed normalized). What is the (normalized) density operator ρ' after a measurement with result a_n? Express your answer in terms of the original density operator ρ. Do not assume the eigenvalue a_n is nondegenerate.

Let ρ be the original density matrix, expressed in terms of pure states and probabilities by

$$\rho = \sum_i f_i |\psi_i\rangle\langle\psi_i| \tag{5}$$

Suppose we have a large number N of systems. Then

$$N_i = N f_i \tag{6}$$

is the number in state $|\psi_i\rangle$.

The number of the N_i systems that have measurement result a_n are

$$N_i' = N_i \langle\psi_i|P_n|\psi_i\rangle. \tag{7}$$

The state of the systems that were in $|\psi_i\rangle$ becomes

$$|\psi_i'\rangle = \frac{P_n|\psi_i\rangle}{\sqrt{\langle\psi_i|P_n|\psi_i\rangle}}, \tag{8}$$

Therefore the total number of systems after the measurement is

$$N' = \sum_i N_i' = \sum_i N_i \langle\psi_i|P_n|\psi_i\rangle. \tag{9}$$

So the probability of being in state $|\psi_i'\rangle$ after the measurement is

$$f_i' = \frac{N_i'}{N'}. \tag{10}$$

But

$$N' = \sum_i N_i \langle\psi_i|P_n|\psi_i\rangle = N \sum_i f_i \langle\psi_i|P_n|\psi_i\rangle = N\mathrm{Tr}(\rho P_n). \tag{11}$$

So

$$
\begin{aligned}
\rho' &= \sum_i f_i' |\psi_i'\rangle\langle\psi_i'| && (12) \\
&= \sum_i \frac{N_i'}{N'} \frac{P_n|\psi_i\rangle\langle\psi_i|P_n}{\langle\psi_i|P_n|\psi_i\rangle} && (13) \\
&= \sum_i \frac{N_i \cancel{\langle\psi_i|P_n|\psi_i\rangle}}{N'} \frac{P_n|\psi_i\rangle\langle\psi_i|P_n}{i\cancel{\langle\psi_i|P_n|\psi_i\rangle}} && (14) \\
&= \frac{N}{N\mathrm{Tr}(\rho P_n)} \sum_i f_i P_n|\psi_i\rangle\langle\psi_i|P_n && (15) \\
&= \frac{P_n \rho P_n}{\mathrm{Tr}(\rho P_n)} && (16)
\end{aligned}
$$

2. Find the linear combination of eigenkets of the S_z operator, $|+\rangle$ and $|-\rangle$, that maximize the uncertainty in $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle$. Show that for the combination you find that the uncertainty relation is not violated.

   In terms of $|+\rangle$ and $|-\rangle$, S_x and S_y can be written

$$
\begin{aligned}
S_x &= \frac{\hbar}{2}\left(|-\rangle\langle+| + |+\rangle\langle-|\right) && (17) \\
S_y &= i\frac{\hbar}{2}\left(|-\rangle\langle+| - |+\rangle\langle-|\right) && (18)
\end{aligned}
$$

   Since we can choose the overall phase, an arbitrary normalized state can be written as

$$
|\alpha\rangle = \cos(\beta/2)|+\rangle + \sin(\beta/2)e^{i\delta}|-\rangle \qquad (19)
$$

   Then

$$
\begin{aligned}
\langle S_x\rangle &= \hbar\cos(\beta/2)\sin(\beta/2)\cos\delta && (20) \\
\langle S_y\rangle &= \hbar\cos(\beta/2)\sin(\beta/2)\sin\delta && (21) \\
\langle S_x^2\rangle &= \frac{\hbar^2}{4} && (22) \\
\langle S_y^2\rangle &= \frac{\hbar^2}{4} && (23) \\
\langle(\Delta S_x)^2\rangle &= \frac{\hbar^2}{4} - \hbar^2\cos^2(\beta/2)\sin^2(\beta/2)\cos^2\delta && (24) \\
\langle(\Delta S_y)^2\rangle &= \frac{\hbar^2}{4} - \hbar^2\cos^2(\beta/2)\sin^2(\beta/2)\sin^2\delta && (25)
\end{aligned}
$$

So we have

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \hbar^4\left[\frac{1}{16}-\frac{1}{4}\cos^2(\beta/2)\sin^2(\beta/2)+\cos^4(\beta/2)\sin^4(\beta/2)\cos^2\delta\sin^2\delta\right] \tag{26}$$

$$= \hbar^4\left[\frac{1}{16}-\frac{1}{4}\sin^2(\beta)+\frac{1}{4}\sin^4(\beta)\sin^2(2\delta)\right] \tag{27}$$

The last term is positive definite, and maximal at $\delta = \frac{\pi}{4}$, or $\sin(2\delta) = 1$.

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \hbar^4\left[\frac{1}{16}-\frac{1}{4}\sin^2\beta+\frac{1}{4}\sin^4\beta\right] \tag{28}$$

$$= \left[\hbar^2\left(\frac{1}{4}-\sin^2\beta\right)\right]^2 \tag{29}$$

The maximum occurs when $\sin\beta = 0 \implies \beta = 0, \pi$

I.e., when $|\alpha\rangle$ is one of the eigenstates of S_z

The uncertainty relation then states

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16} \tag{30}$$

$$\leq \frac{1}{4}\langle[S_x, S_y]\rangle^2 \tag{31}$$

$$= \frac{1}{4}\langle\hbar S_z\rangle^2 \tag{32}$$

$$= \frac{1}{4}\left(\frac{\hbar^2}{2}\right)^2 \tag{33}$$

The inequality holds (and is saturated).

3. A three-dimensional ket space is spanned by an orthonormal basis $|1\rangle$, $|2\rangle$, and $|3\rangle$. In this basis, the operators A and B are represented by

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}. \tag{34}$$

(a) Do A or B have degenerate eigenvalues?

By inspection, the eigenvalues of A are a, $-a$, and $-a$, and there is a degeneracy.

B can be written as

$$B = \left(\begin{array}{c|c} b & \\ \hline & b\sigma_2 \end{array}\right), \tag{35}$$

with σ_2 a Pauli matrix, which has eigenvalues of ± 1.

$\implies$ B has eigenvalues b, b, and $-b$, and is also degenerate.

(b) Show that A and B commute.

One can multiply the matrices explicitly, or note the following:

A and B can be written as

$$A = a\left(\begin{array}{c|c} \mathbb{1}_1 & \\ \hline & -\mathbb{1}_2 \end{array}\right), \quad B = b\left(\begin{array}{c|c} \mathbb{1}_1 & \\ \hline & \sigma_2 \end{array}\right) \tag{36}$$

where $\mathbb{1}_1$ and $\mathbb{1}_2$ are 1- and 2-dimensional identity matrices, respectively.

$$\implies [A, B] = ab\left(\begin{array}{c|c} [\mathbb{1}_1, \mathbb{1}_1] & \\ \hline & [-\mathbb{1}_2, \sigma_2] \end{array}\right) = 0 \tag{37}$$

(c) Find a new set of kets that are the simultaneous eigenkets of A and B. Specify the eigenvalues of A and B for each ket. Are these eigenvalues sufficient to uniquely specify each ket?

The eigenvector of A corresponding to the nondegenerate eigenvalue of A is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle \tag{38}$$

.

$|1\rangle$ is also an eigenvector of B, so we choose the first vector in our simultaneous basis:

$$|1'\rangle = |1\rangle \tag{39}$$

Any vectors in the remaining subspace will automatically be eigenkets of A. We want to choose the two orthogonal vectors within this space that are eigenvectors of B, and we will then have simultaneous eigenvectors.

$\implies$ We need to diagonalize σ_2 within the $|2\rangle$, $|3\rangle$ subspace. $\implies$ we should choose

$$|2'\rangle = \frac{1}{\sqrt{2}}(|2\rangle + i|3\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix} \tag{40}$$

$$|3'\rangle = \frac{1}{\sqrt{2}}(|2\rangle - i|3\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\end{pmatrix} \tag{41}$$

These are now simultaneous eigenvectors of A and B, with eigenvalues

$$|1'\rangle = |a, b\rangle \tag{42}$$

$$|2'\rangle = |-a, b\rangle \tag{43}$$

$$|3'\rangle = |-a, -a\rangle \tag{44}$$

The degeneracy is completely broken, and the set of eigenvalues of A and B, is sufficient to specify each state.

4. Let $f(\mathbf{x})$ be an arbitrary function of $\mathbf{x}$ and $g(\mathbf{p})$ an arbitrary function of $\mathbf{p}$. Evaluate the commutators, $[p_i, f(\mathbf{x})]$ and $[x_i, g(\mathbf{p})]$

4.4. $[p_i, f(\vec{x})] = p_i f(\vec{x}) - f(\vec{x}) p_i$

$$\psi(\vec{x}) \xrightarrow{f(\vec{x})} f(\vec{x})\psi(\vec{x}) \xrightarrow{p_i = -i\hbar\, \partial/\partial x_i} -i\hbar\left[\frac{\partial f}{\partial x_i}\psi + f\frac{\partial \psi}{\partial x_i}\right]$$

$$\psi(\vec{x}) \xrightarrow{p_i} -i\hbar\frac{\partial \psi}{\partial x_i} \xrightarrow{f(\vec{x})} -i\hbar\, f(\vec{x})\frac{\partial \psi}{\partial x_i}$$

subt, $-i\hbar\dfrac{\partial f}{\partial x_i}\psi$

$$\Rightarrow \boxed{[p_i, f(\vec{x})] = -i\hbar\frac{\partial f}{\partial x_i}}$$

For other commutator use momentum wave fns.

$$\phi(\vec{p}) \xrightarrow{g(\vec{p})} g(\vec{p})\phi(\vec{p}) \xrightarrow{x_i = i\hbar\, \frac{\partial}{\partial p_i}} i\hbar\left[\frac{\partial g}{\partial p_i}\phi + g\frac{\partial \phi}{\partial p_i}\right]$$

$$\phi(\vec{p}) \xrightarrow{x_i} i\hbar\frac{\partial \phi}{\partial p_i} \xrightarrow{g(\vec{p})} i\hbar\, g(\vec{p})\frac{\partial \phi}{\partial p_i}$$

subt, get $i\hbar\dfrac{\partial g}{\partial p_i}\phi$

$\Rightarrow$

~~[pi, g~~

$$\boxed{[x_i, g(\vec{p})] = i\hbar\frac{\partial g}{\partial p_i}}$$