Soluções:

FisMat I Exercicios 1

Prazo: 30 agosto 2021

1. Determine a série de Fourier da função periódica (descontínua)

$$f(x) = \cos(px) \tag{1}$$

com $-\pi \leq x < \pi$ e com $p \notin \mathbb{Z}$. (I.e., $p \neq 0, \pm 1, \pm 2, \ldots$).

A partir do resultado mostre que

$$\frac{\pi}{4} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1} \tag{2}$$

O período da função é 2π. Os coeficientes cosenos são

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(px)dx && (3)\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\cos\left[(n+p)x\right] + \cos\left[(n-p)x\right]\Big)dx && (4)\\
&= \frac{1}{2\pi}\frac{1}{n+p}\,\text{sen}\left[(n+p)x\right]\Big|_{-\pi}^{\pi} + \frac{1}{2\pi}\frac{1}{n-p}\,\text{sen}\left[(n-p)x\right]\Big|_{-\pi}^{\pi} && (5)\\
&= \frac{1}{\pi}\frac{1}{n+p}\,\text{sen}\left[(n+p)\pi\right] + \frac{1}{\pi}\frac{1}{n-p}\,\text{sen}\left[(n-p)\pi\right] && (6)\\
&= \frac{1}{\pi(n^2-p^2)}\Big((n-p)\,\text{sen}\left[(n+p)\pi\right] + (n+p)\,\text{sen}\left[(n-p)\pi\right]\Big) && (7)\\
&= \frac{1}{2\pi(n^2-p^2)}\left[n\cos(p\pi)\,\text{sen}(n\pi) - p\cos(n\pi)\,\text{sen}(p\pi)\right] && (8)\\
&= \frac{(-1)^n p}{2\pi(p^2-n^2)}\,\text{sen}(p\pi) && (9)
\end{aligned}$$

Note-se que $\text{sen}(n\pi) = 0$ e $\cos(n\pi) = (-1)^n, \quad \forall n \in \mathbb{Z}$.

E os senos:

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \text{sen}(nx)\cos(px)dx && (10)\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\text{sen}\,[(n+p)x] + \text{sen}\,[(n-p)x]\Big)dx && (11)\\
&= -\frac{1}{\pi}\frac{1}{n+p}\cos\,[(n+p)x]\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\frac{1}{n-p}\cos\,[(n-p)x]\Big|_{-\pi}^{\pi} && (12)\\
&= 0 && (13)
\end{aligned}
$$

A série inteira é

$$
\begin{aligned}
\cos(px) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) && (14)\\
&= \frac{\text{sen}(p\pi)}{p\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n p\,\text{sen}(p\pi)}{p^2 - n^2}\cos(nx) && (15)
\end{aligned}
$$

Note-se que $\text{sen}(n\pi) = 0, \quad \forall n \in \mathbb{Z}$.

Para $p = \frac{1}{4}$, temos

$$
\begin{aligned}
\cos(x/4) &= \frac{4\,\text{sen}(\pi/4)}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{4(-1)^n\,\text{sen}(\pi/4)}{(1^2 - 16n^2)}\cos(nx) && (16)\\
&= \frac{4}{\sqrt{2}\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{4(-1)^n}{\sqrt{2}(1^2 - 16n^2)}\cos(nx) && (17)
\end{aligned}
$$

No ponto $x = \pi$, a função é

$$
\cos(x/4) = \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}\pi} + \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2}(1 - 16n^2)}(-1)^n \qquad (18)
$$

Multiplicando por $\sqrt{2}\pi/4$,

$$\frac{\pi}{4} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1} \tag{19}$$

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2. A variação de pressão $f(t) = \Delta P/P_0$ em torno da pressão atmosférica, que corresponde a uma onda sonora que se propaga no ar é dada por

$$f(t) = \begin{cases} 1 & \text{para } 0 \leq t < \frac{T}{4} \\ -\frac{7}{8} & \text{para } \frac{T}{4} \leq t < \frac{T}{2} \\ \frac{7}{8} & \text{para } \frac{T}{2} \leq t < \frac{3T}{4} \\ -1 & \text{para } \frac{3T}{4} \leq t < T \end{cases} \tag{20}$$

onde T é o período.

Considerando a decomposição dessa onda nos seus harmônicos, que harmônico pode ser detectado com mais clareza? Dê a intensidade relativa ($\sim b_n^2/b_1^2$ onde b_n é o n-ésimo coeficiente da série de Fourier) de cada um dos 10 primeiros harmônicos.

As coeficientes são

$$b_n = \frac{2}{T}\int_0^T f(t)\,\text{sen}(n\frac{2\pi t}{T})dt \tag{21}$$

$$= \frac{2}{T}\left[\int_0^{\frac{T}{4}} \text{sen}(n\frac{2\pi t}{T})dt - \frac{7}{8}\int_{\frac{T}{4}}^{\frac{T}{2}} \text{sen}(n\frac{2\pi t}{T})dt + \frac{7}{8}\int_{\frac{T}{2}}^{\frac{3T}{4}} \text{sen}(n\frac{2\pi t}{T})dt - \int_{\frac{3T}{4}}^{T} \text{sen}(n\frac{2\pi t}{T})dt\right] \tag{22}$$

$$= \frac{1}{n\pi}\Big[-\cos(n\frac{\pi}{2}) + \cos(0) + \frac{7}{8}\cos(n\pi) - \frac{7}{8}\cos(n\frac{\pi}{2}) \tag{23}$$

$$- \frac{7}{8}\cos(n\frac{3\pi}{2}) + \frac{7}{8}\cos(n\pi) + \cos(n2\pi) - \cos(n\frac{3\pi}{2})\Big] \tag{24}$$

$$= \frac{1}{n\pi}\left[1 - \frac{15}{8}\cos(n\frac{\pi}{2}) + \frac{7}{4}\cos(n\pi) - \frac{15}{8}\cos(n\frac{3\pi}{2}) + \cos(n2\pi)\right] \tag{25}$$

$$= \frac{1}{n\pi}\left[2 + \frac{7}{4}(-1)^n + \frac{15}{4}\left[(-1)^{\frac{3n}{2}} + (-1)^{\frac{n}{2}}\right]\right] \tag{26}$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\frac{2\pi t}{T})dt && (27)\\
&= \frac{2}{T}\left[\int_0^{\frac{T}{4}}\cos(n\frac{2\pi t}{T})dt - \frac{7}{8}\int_{\frac{T}{4}}^{\frac{T}{2}}\cos(n\frac{2\pi t}{T})dt + \frac{7}{8}\int_{\frac{T}{2}}^{\frac{3T}{4}}\cos(n\frac{2\pi t}{T})dt - \int_{\frac{3T}{4}}^{T}\cos(n\frac{2\pi t}{T})dt\right] && (28)\\
&= \frac{1}{n\pi}\left[\text{sen}(n\frac{\pi}{2}) - \text{sen}(0) - \frac{7}{8}\text{sen}(n\pi) + \frac{7}{8}\text{sen}(n\frac{\pi}{2})\right. && (29)\\
&\quad \left. + \frac{7}{8}\text{sen}(n\frac{3\pi}{2}) - \frac{7}{8}\text{sen}(n\pi) - \text{sen}(n2\pi) + \text{sen}(n\frac{3\pi}{2})\right] && (30)\\
&= \frac{1}{n\pi}\left[\frac{15}{8}\text{sen}(n\frac{\pi}{2}) - \frac{14}{8}\text{sen}(n\pi) + \frac{15}{8}\text{sen}(n\frac{3\pi}{2}) - \text{sen}(n2\pi)\right] && (31)\\
&= \frac{1}{n\pi}\frac{15}{8}\left[\text{sen}(n\frac{\pi}{2}) + \text{sen}(n\frac{3\pi}{2})\right] && (32)\\
&= 0 && (33)
\end{aligned}
$$

Então,

$$b_1 = 2 - \frac{7}{4} = \frac{1}{4} \tag{34}$$

$$\frac{b_2^2}{b_1^2} = 4^2 \left[2 + \frac{7}{4} - \frac{15}{2}\right]^2 = 225 \tag{35}$$

$$\frac{b_3^2}{b_1^2} = \frac{1}{9} \tag{36}$$

$$\frac{b_4^2}{b_1^2} = 0 \tag{37}$$

$$\frac{b_5^2}{b_1^2} = \frac{1}{25} \tag{38}$$

$$\frac{b_6^2}{b_1^2} = 25 \tag{39}$$

$$\frac{b_7^2}{b_1^2} = \frac{1}{49} \tag{40}$$

$$\frac{b_8^2}{b_1^2} = 0 \tag{41}$$

$$\frac{b_9^2}{b_1^2} = \frac{1}{81} \tag{42}$$

$$\frac{b_{10}^2}{b_1^2} = 9 \tag{43}$$

$$\vdots \tag{44}$$

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3. Determine a série de Fourier que corresponde à função

$$f(x) = \begin{cases} 2x & \text{para } 0 \leq x \leq 3 \\ 0 & \text{para } -3 < x < 0 \end{cases} \tag{45}$$

cujo período é igual a 6.

$$a_n = \frac{1}{3} \int_{-3}^{3} dx f(x) \cos\left(\frac{n\pi x}{3}\right) \tag{46}$$

$$= \frac{2}{3} \int_{0}^{3} dx x \cos\left(\frac{n\pi x}{3}\right) \tag{47}$$

Integrando por partes:

$$a_n = \frac{2}{3} x \frac{3\,\text{sen}\left(\frac{n\pi x}{3}\right)}{n\pi}\bigg|_0^3 - \frac{2}{3}\int_0^3 dx \frac{3\,\text{sen}\left(\frac{n\pi x}{3}\right)}{n\pi} \tag{48}$$

$$= 0 + \frac{6}{(n\pi)^2}\cos\left(\frac{n\pi x}{3}\right)\bigg|_0^3 \tag{49}$$

$$= \frac{6}{(n\pi)^2}\left[\cos(n\pi) - 1\right] \tag{50}$$

$$= \frac{6}{(n\pi)^2}\left[(-1)^n - 1\right] \tag{51}$$

$$= \begin{cases} 0 & n \text{ par} \\ \frac{-12}{(n\pi)^2} & n \text{ impar} \end{cases} \tag{52}$$

Semelhantemente

$$b_n = \frac{2}{3}\int_0^3 dx x\,\text{sen}\left(\frac{n\pi x}{3}\right) \tag{53}$$

$$= \frac{-6}{n\pi}\cos(n\pi) \tag{54}$$

$$= \frac{-6}{n\pi}(-1)^n \tag{55}$$

E finalmente a série é

$$f(x) = \frac{3}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos\left(\frac{n\pi x}{3}\right) - \frac{(-1)^n}{n}\,\text{sen}\left(\frac{n\pi x}{3}\right)\right] \tag{56}$$

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4. Obtenha a série de Fourier de senos que representa a função $f(x) = e^x$ no intervalo $(0, \pi)$.

$$a_n = \frac{2}{\pi}\int_0^\pi dx f(x)\,\text{sen}\,(nx) \tag{57}$$
$$= \frac{2}{\pi}\int_0^\pi dx e^x\,\text{sen}\,(nx) \tag{58}$$
$$= \frac{2}{\pi}\int_0^\pi dx e^x \frac{e^{inx} - e^{-inx}}{2i} \tag{59}$$
$$= \frac{i}{\pi}\int_0^\pi dx \left(e^{(1-in)x} - e^{(1+in)x}\right) \tag{60}$$
$$= \frac{i}{\pi}\left[\frac{1}{1-in}e^{(1-in)x}\Big|_0^\pi - \frac{1}{1+in}e^{(1+in)x}\Big|_0^\pi\right] \tag{61}$$
$$= \frac{i}{\pi}\left[\frac{1}{1-in}\left(e^{(1-in)\pi} - 1\right) - \frac{1}{1+in}\left(e^{(1+in)\pi} - 1\right)\right] \tag{62}$$
$$= \frac{i}{\pi}\left[\frac{1}{1-in}\left(e^{\pi}(-1)^n - 1\right) - \frac{1}{1+in}\left(e^{\pi}(-1)^n - 1\right)\right] \tag{63}$$
$$= \frac{i}{\pi}\left(e^{\pi}(-1)^n - 1\right)\left(\frac{1}{1-in} - \frac{1}{1+in}\right) \tag{64}$$
$$= \frac{i}{\pi}\left(e^{\pi}(-1)^n - 1\right)\frac{2in}{1+n^2} \tag{65}$$
$$= \frac{2n}{\pi(1+n^2)}\left[1 - e^{\pi}(-1)^n\right] \tag{66}$$

Então, a série de senos é

$$g_s(x) = \sum_{n=1}^{\infty}\frac{2n}{\pi(1+n^2)}\left[1 - e^{\pi}(-1)^n\right]\text{sen}\,nx \tag{67}$$
$$= \frac{2}{\pi}\left[\frac{1+e^\pi}{2}\,\text{sen}\,x + \frac{2(1-e^\pi)}{5}\,\text{sen}\,2x + \frac{3(1+e^\pi)}{10}\,\text{sen}\,3x + \ldots\right] \tag{68}$$

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5. Obtenha a série de Fourier de co-senos que representa a função $f(x) = e^x$ no intervalo $(0, \pi)$.

Como o exercício anterior:

$$b_n = \frac{2}{\pi}\int_0^\pi dx f(x)\cos(nx) \tag{69}$$
$$= \frac{2}{\pi}\int_0^\pi dx e^x \cos(nx) \tag{70}$$
$$= \frac{2}{\pi}\int_0^\pi dx e^x \frac{e^{inx}+e^{-inx}}{2} \tag{71}$$
$$= \frac{1}{\pi}\int_0^\pi dx\left(e^{(1-in)x}+e^{(1+in)x}\right) \tag{72}$$
$$= \frac{1}{\pi}\left[\frac{1}{1-in}e^{(1-in)x}\Big|_0^\pi + \frac{1}{1+in}e^{(1+in)x}\Big|_0^\pi\right] \tag{73}$$
$$= \frac{1}{\pi}\left[\frac{1}{1-in}\left(e^{(1-in)\pi}-1\right)+\frac{1}{1+in}\left(e^{(1+in)\pi}-1\right)\right] \tag{74}$$
$$= \frac{1}{\pi}\left[\frac{1}{1-in}\left(e^{\pi}(-1)^n-1\right)+\frac{1}{1+in}\left(e^{\pi}(-1)^n-1\right)\right] \tag{75}$$
$$= \frac{1}{\pi}\left(e^{\pi}(-1)^n-1\right)\left(\frac{1}{1-in}+\frac{1}{1+in}\right) \tag{76}$$
$$= \frac{1}{\pi}\left(e^{\pi}(-1)^n-1\right)\frac{2}{1+n^2} \tag{77}$$
$$= \frac{2}{\pi(1+n^2)}\left[e^{\pi}(-1)^n-1\right] \tag{78}$$

e a série de co-senos é

$$g_c(x) = \frac{1}{\pi}(e^\pi-1)+\sum_{n=1}^{\infty}\frac{2}{\pi(1+n^2)}\left[e^{\pi}(-1)^n-1\right]\cos nx \tag{79}$$
$$= \frac{2}{\pi}\left[\frac{e^\pi-1}{2}-\frac{e^\pi+1}{2}\cos x+\frac{e^\pi+1}{5}\cos 2x - -\frac{e^\pi+1}{10}\cos 3x\ldots\right] \tag{80}$$

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6. Ache a série de Fourier da função

$$f(x) = \begin{cases} 0 & \text{para } -5 < x < 0 \\ 3 & \text{para } 0 < x < 5 \end{cases} \tag{81}$$

cujo período é igual a 10.

$$a_n = \frac{1}{5}\int_0^5 dx 3\cos\left(\frac{n\pi x}{5}\right) \tag{82}$$
$$= 3\delta_{n,0} \tag{83}$$

$$b_n = \frac{1}{5}\int_0^5 dx 3\,\text{sen}\left(\frac{n\pi x}{5}\right) \tag{84}$$
$$= -\frac{3}{5}\frac{5}{n\pi}\left[\cos n\pi - 1\right] \tag{85}$$
$$= -\frac{3}{5}\frac{5}{n\pi}\left[(-1)^n - 1\right] \tag{86}$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3}{n\pi}\left[1-(-1)^n\right]\text{sen}\, nx \tag{87}$$
$$= \frac{3}{2} + \frac{6}{\pi}\sum_{n=0}^{\infty}\frac{\text{sen}\left[(2n+1)x\right]}{2n+1} \tag{88}$$

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7. Uma corrente alternada $i(t) = A\,\text{sen}(\omega t)$ passou por a) um retificador de meia onda que transmite a corrente somente quando ela passa no sentido positivo e b) um retificador de onda completa, que transmite o valor absoluto (instantâneo) da corrente. Mostre que no primeiro caso a saída é

$$\frac{A}{\pi} + \frac{A}{2}\text{sen}(\omega t) - \frac{2A}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\cos[(n+1)\omega t]}{n(n+2)} \tag{89}$$

e no segundo caso é

$$\frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos(n\omega t)}{n^2-1} \tag{90}$$

(a) A corrente transmitida é

$$i(t) = \begin{cases} A \,\mathrm{sen}\, \omega t, & 0 \leq t \leq \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega} \end{cases} \tag{91}$$

As coeficientes de Fourier são

$$a_n = \frac{\omega}{\pi} \int_0^{\pi/\omega} dt A \,\mathrm{sen}\, \omega t \cos(n\omega t) \tag{92}$$

$$= A \frac{1 + \cos n\pi}{\pi(1 - n^2)} \tag{93}$$

$$= A \frac{1 + (-1)^n}{\pi(1 - n^2)} \tag{94}$$

$$b_n = \frac{\omega}{\pi} \int_0^{\pi/\omega} dt A \,\mathrm{sen}\, \omega t \,\mathrm{sen}\,(n\omega t) \tag{95}$$

$$= \frac{A}{2} \delta_{n,1} \tag{96}$$

A série é

$$i(t) = \frac{A}{\pi} + \frac{A}{2} \mathrm{sen}(\omega t) - \frac{2A}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos[(n)\omega t]}{1 - n^2} \tag{97}$$

$$= \frac{A}{\pi} + \frac{A}{2} \mathrm{sen}(\omega t) - \frac{2A}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos[(n+1)\omega t]}{n(n+2)} \tag{98}$$

(b) A corrente transmitida é

$$i(t) = |A \,\mathrm{sen}(\omega t)| \tag{99}$$

$$= \begin{cases} |A| \,\mathrm{sen}\, \omega t, & 0 \leq t \leq \frac{\pi}{\omega} \\ -|A| \,\mathrm{sen}\, \omega t, & \frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega} \end{cases} \tag{100}$$

Como a função é simétrica, somente tem coeficientes de co-seno. As coeficientes são

$$a_n = \frac{2\omega}{\pi}\int_0^{\pi/\omega} dt A \operatorname{sen}\omega t \cos(n\omega t) \tag{101}$$
$$= 2A\frac{1+\cos n\pi}{\pi(1-n^2)} \tag{102}$$
$$= 2A\frac{1+(-1)^n}{\pi(1-n^2)} \tag{103}$$

A série é

$$i(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos(n\omega t)}{n^2-1} \tag{104}$$

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8. Utilizando a equação de Parseval, e considerando a função

$$f(x) = \begin{cases} -1 & \text{se } -\pi < x < 0 \\ 1 & \text{se } 0 < x < \pi \end{cases} \tag{105}$$

mostre que $1 + 1/3^2 + 1/5^2 + 1/7^2 + \ldots = \pi^2/8$.

A equação de Parseval é

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx f^2(x) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \tag{106}$$

onde

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\cos nx \tag{107}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx f(x)\operatorname{sen} nx \tag{108}$$

Na aula, calculamos os coeficientes de Fourier:

$$a_n = 0 \tag{109}$$

$$b_n = \frac{2}{\pi n}\left[1 - (-1)^n\right] \tag{110}$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases} \tag{111}$$

A equação fica, então

$$\frac{1}{\pi}\int_{-\pi}^{\pi} dx f^2(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} dx = 2 \tag{112}$$

$$= \sum_{n=1,3,5,\ldots}^{\infty} b_n^2 = \frac{16}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} \tag{113}$$

$$= \frac{16}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \tag{114}$$

Então,

$$\sum_{n=0}^{\infty}\frac{1}{2n+1} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8} \tag{115}$$

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9. Considere a função de período $T = 2\pi$, $f(x) = e^x$ para $-\pi < x < \pi$. Obtenha os coeficientes da série complexa de Fourier c_n.

As coeficientes complexas são

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx e^{-inx} f(x) \tag{116}$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} dx e^{-inx+x} \tag{117}$$
$$= \frac{1}{\pi(1-in)} e^{(1-in)x}\Big|_{-\pi}^{\pi} \tag{118}$$
$$= \frac{1}{\pi(1-in)}\left(e^{(1-in)\pi} - e^{-(1-in)\pi}\right) \tag{119}$$
$$= \frac{(-1)^n}{\pi(1-in)}\left(e^{\pi} - e^{-\pi}\right) \tag{120}$$
$$= \frac{(-1)^n \operatorname{senh}\pi}{\pi(1-in)} \tag{121}$$
$$= \frac{(1+in)(-1)^n \operatorname{senh}\pi}{\pi(n^2+1)} \tag{122}$$

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10. Uma corda de comprimento L vibra presa nas suas extremidades $x = 0$ e $x = L$. O movimento é descrito pela equação de onda $\frac{\partial^2 u(t,x)}{\partial t^2} = v^2 \frac{\partial^2 u(t,x)}{\partial x^2}$. Supondo que a solução seja da forma $u(t,x) = \sum_{n=1}^{\infty} b_n(t)\,\operatorname{sen}(n\pi x/L)$, determine os coeficientes $b_n(t)$ considerando as condições iniciais $u(0,x) = f(x)$ e $\partial u(0,x)/\partial t = g(x)$.

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \sum_{n=1}^{\infty} \frac{\partial^2 b_n(t)}{\partial t^2}\operatorname{sen}(n\pi x/L) \tag{123}$$
$$= v^2\frac{\partial^2 u(t,x)}{\partial x^2} = -\sum_{n=1}^{\infty} \frac{v^2 n^2 \pi^2}{L^2} b_n(t)\,\operatorname{sen}(n\pi x/L) \tag{124}$$
$$\Longrightarrow \frac{\partial^2 b_n(t)}{\partial t^2} = \frac{v^2 n^2 \pi^2}{L^2} b_n(t) \tag{125}$$

A solução geral é

$$b_n(t) = A_n \cos\left(\frac{vn\pi t}{L}\right) + B_n \operatorname{sen}\left(\frac{vn\pi t}{L}\right) \tag{126}$$

As condições iniciais são

$$u(0,x) = \sum_{n=1}^{\infty} b_n(0)\ \text{sen}(n\pi x/L) \tag{127}$$

$$= \sum_{n=1}^{\infty} A_n\ \text{sen}(n\pi x/L) \tag{128}$$

$$= f(x) \tag{129}$$

$$\partial u(0,x)/\partial t = \sum_{n=1}^{\infty} b'_n(0)\ \text{sen}(n\pi x/L) \tag{130}$$

$$= \sum_{n=1}^{\infty} B_n \frac{vn\pi}{L}\ \text{sen}(n\pi x/L) \tag{131}$$

$$= g(x) \tag{132}$$

As coeficientes são dados por as coeficientes de Fourier das funções $f(x)$ e $g(x)$:

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\text{sen}(n\pi x/L) \tag{133}$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\,\text{sen}(n\pi x/L) \tag{134}$$

$$B_n = \frac{1}{vn\pi}\int_{-L}^{L} g(x)\,\text{sen}(n\pi x/L) \tag{135}$$

$$= \frac{2}{vn\pi}\int_{0}^{L} g(x)\,\text{sen}(n\pi x/L) \tag{136}$$

e

$$b_n(t) = A_n \cos\left(\frac{vn\pi t}{L}\right) + B_n\,\text{sen}\left(\frac{vn\pi t}{L}\right) \tag{137}$$
