Propriedades da Transformada de Laplace

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty dt\ F(t) e^{-st}$$

- Se $\mathcal{L}\{F(t)\} = f(s)$ então

$$\mathcal{L}\left\{e^{at}F(t)\right\} = \int_0^\infty e^{at}e^{-st}F(t) = \int_0^\infty e^{-(s-a)t}F(t) = f(s-a)$$

- Assim, como

$$\mathcal{L}\{\text{sen}\ kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{e^{at}\,\text{sen}\ kt\} = \frac{k}{(s-a)^2+k^2}$$

Trocamos $s \to s-a$ e estamos prontos!

- Exemplo: Oscilador harmônico amortecido

$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = 0$$

com condições iniciais

$$x(0^+) = x_0$$
$$\dot{x}(0^+) = 0.$$

Aplicando a transformada obtemos

$$s^2 \tilde{x}(s) - s x(0^+) - \cancelto{0}{\dot{x}(0^+)} + \gamma s \tilde{x}(s) - \gamma x(0^+) + \omega_0^2 \tilde{x}(s) = 0.$$

Então

$$\tilde{x}(s) = \frac{x_0(s+\gamma)}{s(s+\gamma) + \omega_0^2}$$

Exemplo: Oscilador harmônico amortecido

$$\tilde{x}(s) = \frac{x_0(s+\gamma)}{s(s+\gamma)+\omega_0^2}$$

- Completando o quadrado,

$$\begin{aligned} s(s+\gamma)+\omega_0^2 &= s^2 + s\gamma + \frac{\gamma^2}{4} - \frac{\gamma^2}{4} + \omega_0^2 \\ &= \left(s+\frac{\gamma}{2}\right)^2 + \left(\omega_0^2 - \frac{\gamma^2}{4}\right) \end{aligned}$$

e

$$\begin{aligned} \tilde{x}(s) &= \frac{x_0(s+\frac{\gamma}{2})}{s(s+\gamma)+\omega_0^2} + \frac{x_0\frac{\gamma}{2}}{s(s+\gamma)+\omega_0^2} \\ &= \frac{x_0(s+\frac{\gamma}{2})}{\left(s+\frac{\gamma}{2}\right)^2 + \left(\omega_0^2 - \frac{\gamma^2}{4}\right)} + \frac{x_0\frac{\gamma}{2}}{\left(s+\frac{\gamma}{2}\right)^2 + \left(\omega_0^2 - \frac{\gamma^2}{4}\right)} \end{aligned}$$

Define $\omega_1^2 \equiv \omega_0^2 - \frac{\gamma^2}{4} > 0$ e assim

Exemplo: Oscilado harmônico amortecido

$$\tilde{x}(s) = \frac{x_0\left(s + \frac{\gamma}{2}\right)}{\left(s + \frac{\gamma}{2}\right)^2 + \left(\omega_0^2 - \frac{\gamma^2}{4}\right)} + \frac{x_0 \frac{\gamma}{2}}{\left(s + \frac{\gamma}{2}\right)^2 + \left(\omega_0^2 - \frac{\gamma^2}{4}\right)}$$

- Define $\omega_1^2 \equiv \omega_0^2 - \frac{\gamma^2}{4} > 0$ e assim

$$\tilde{x}(s) = \frac{x_0\left(s + \frac{\gamma}{2}\right)}{\left(s + \frac{\gamma}{2}\right)^2 + \omega_1^2} + \frac{x_0 \gamma}{2\omega_1} \frac{\omega_1}{\left(s + \frac{\gamma}{2}\right)^2 + \omega_1^2} x(t) \quad = \mathcal{L}^{-1}\left\{\tilde{x}(s)\right\}$$

e como

$$\mathcal{L}\left\{\cos(\omega_1 t) e^{-\frac{\gamma}{2} t}\right\} = \frac{s + \frac{\gamma}{2}}{\left(s + \frac{\gamma}{2}\right)^2 + \omega_1^2}$$

$$\mathcal{L}\left\{\text{sen}(\omega_1 t) e^{-\frac{\gamma}{2} t}\right\} = \frac{\omega_1}{\left(s + \frac{\gamma}{2}\right)^2 + \omega_1^2}$$

vemos que $x(t) = x_0 e^{-\frac{\gamma}{2} t}\left[\cos \omega_1 t + \frac{\gamma}{2\omega_1} \text{sen}\, \omega_1 t\right]$

- Quando $\gamma \to 0$ nós recuperamos o oscilador harmônico.

Exemplo: Oscilado harmônico amortecido

- Define $\omega_1^2 \equiv \omega_0^2 - \frac{\gamma^2}{4} > 0$ e assim

$$\tilde{x}(s) = \frac{x_0(s+\frac{\gamma}{2})}{\left(s+\frac{\gamma}{2}\right)^2 + \omega_1^2} + \frac{x_0\gamma}{2\omega_1}\frac{\omega_1}{\left(s+\frac{\gamma}{2}\right)^2 + \omega_1^2} x(t) \quad = \mathcal{L}^{-1}\left\{\tilde{x}(s)\right\}$$

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Translação em t

- Seja

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty dt e^{-st} F(t).$$

- Considere agora

$$e^{-sb} f(s) = \int_0^\infty dt e^{-s(b+t)} F(t)$$

com $b > 0$. Substituindo variáveis $b + t = z, dt = dz$,

$$\begin{aligned} e^{-sb} f(s) &= \int_b^\infty dz e^{-sz} F(z-b) \\ &= \int_0^\infty dz e^{-sz} F(z-b)\theta(z-b) \end{aligned}$$

- Então,

$$\mathcal{L}\left\{F(t-b)\theta(t-b)\right\} = e^{-sb} f(s)$$

Derivada de uma transformada

- Supondo que $F(t)$ seja contínua por partes e o domínio de s é tal que $e^{-st}F(t)$ converge exponencialmente, a integral que define a transformada

$$f(s) = \int_0^\infty e^{-st}F(t)$$

converge uniformemente e pode ser derivada com relação a s.

- Então,

$$\begin{aligned}\frac{d}{ds}f(s) &= \int_0^\infty dt(-t)e^{-st}F(t)\\ &= \mathcal{L}\{-tF(t)\}\end{aligned}$$

e assim

$$\frac{d^n}{ds^n}f(s) = \mathcal{L}\{(-t)^n F(t)\}$$

- Todas as integrais serão uniformemente convergentes por causa do comportamento de $e^{-st}F(t)$ quando $t \to \infty$.

Derivada de uma transformada

$$\frac{d^n}{ds^n} f(s) = \mathcal{L}\{(-t)^n F(t)\}$$

- Exemplo:

$$\mathcal{L}\left\{e^{kt}\right\} = \int_0^\infty dt e^{-st} e^{kt} = \frac{1}{s-k}, \qquad s > k$$

$$\mathcal{L}\left\{te^{kt}\right\} = -\frac{d}{ds}\frac{1}{s-k} = \frac{1}{(s-k)^2}, \qquad s > k$$

Integrais de transformadas de Laplace

- Se $F(t)$ for contínua por pedaços e s grande o suficiente de forma que $e^{-st}F(t)$ decresça exponencialmente para $t \to \infty$, a integral

$$f(x) = \int_0^\infty e^{-xt} F(t)$$

  é uniformemente convergente com respeito a x.
- Assim, podemos trocar a ordem das integrais da equação abaixo:

$$\int_s^b f(x)dx = \int_s^b dx \int_0^\infty dt e^{-xt} F(t) = \int_0^\infty dt F(t) \int_s^b dx e^{-xt}$$
$$= \int_0^\infty dt F(t) \left(-\frac{1}{t}\right) \left(e^{-tb} - e^{-ts}\right)$$

- Agora, tomamos $b \to \infty$

$$\int_s^\infty dx f(x) = \int_0^\infty dt \frac{F(t)}{t} e^{-st} = \mathcal{L}\left\{\frac{F(t)}{t}\right\}$$

- Se $F(t)/t$ é finito em $t = 0$ ou diverge mais fracamente do que $\sim 1/t$, então $\mathcal{L}\{F(t)/t\}$ existe.

Convolução

- Seja $\quad f_1(s) = \mathcal{L}\{F_1(t)\}, \quad f_2(s) = \mathcal{L}\{F_2(t)\}$
- Então, vamos mostrar que

$$\mathcal{L}\left\{\int_0^t du F_1(t-u)F_2(u)\right\} = f_1(s)f_2(s) = \mathcal{L}\{F_1(t)\}\mathcal{L}\{F_2(t)\}$$

- De fato,

$$\mathcal{L}\left\{\int_0^t du F_1(t-u)F_2(u)\right\} = \int_0^\infty dt e^{-st}\int_0^t du F_1(t-u)F_2(u)$$

$$= \int_0^\infty dt e^{-st}\int_0^\infty du\theta(t-u)F_1(t-u)F_2(u)$$

$$= \int_0^\infty du F_2(u)\int_0^\infty dt e^{-st}\theta(t-u)F_1(t-u)$$

$$= \int_0^\infty du F_2(u)e^{-su}f_1(s) = f_1(s)\int_0^\infty du e^{-st}F_2(u)$$

$$= f_1(s)f_2(s)$$

Convolução

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- Então, vamos mostrar que

$$\mathcal{L}\left\{\int_0^t du F_1(t-u)F_2(u)\right\} = f_1(s)f_2(s) = \mathcal{L}\{F_1(t)\}\mathcal{L}\{F_2(t)\}$$

- Exemplo:

$$\mathcal{L}\left\{\int_0^t du F(u)\right\} = \mathcal{L}\left\{\int_0^t du F_1(t-u)F_2(u)\right\}$$

com $\quad F_1(t) = 1, \quad F_2(t) = F(t)$

- Então,

$$\begin{aligned}\mathcal{L}\left\{\int_0^t du F(u)\right\} &= f_1(s)f_2(s) \\ &= \frac{1}{s}\mathcal{L}\{F(t)\} \\ &= \frac{f(s)}{s}\end{aligned}$$