Lembrete

- Uma função pode ser representada por sua série de Fourier

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \operatorname{sen}(nx)]$$

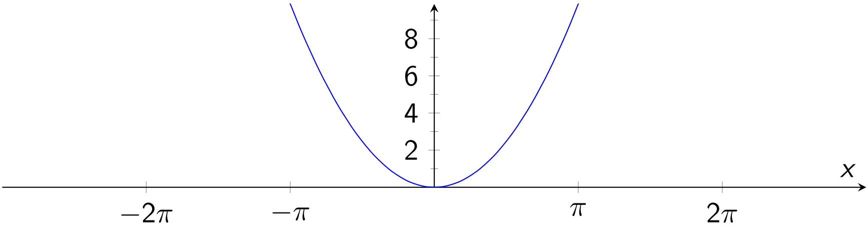
com coeficientes

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \; f(x) \cos mx, \qquad m \geq 0$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \; f(x) \operatorname{sen} mx, \qquad m > 0$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$

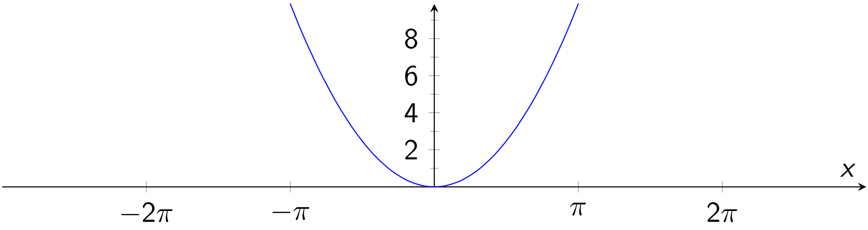


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ f(x)\,\text{sen}\, nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 = \frac{2}{3}\pi^2$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$

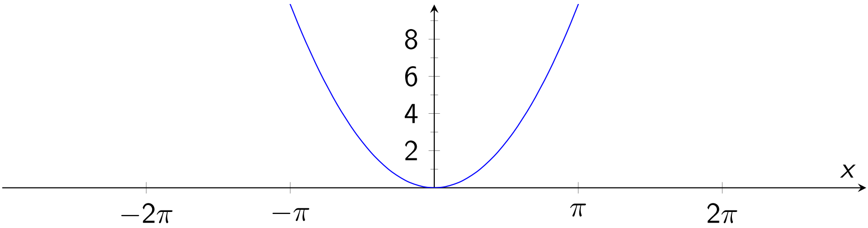


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ f(x)\,\text{sen}\,nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 = \frac{2}{3}\pi^2$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$

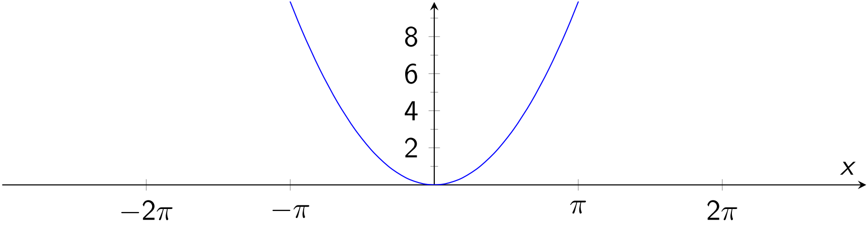


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \underbrace{x^2}_{\text{par}} \underbrace{\text{sen}\, nx}_{\text{impar}} = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, x^2 = \frac{2}{3}\pi^2$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$

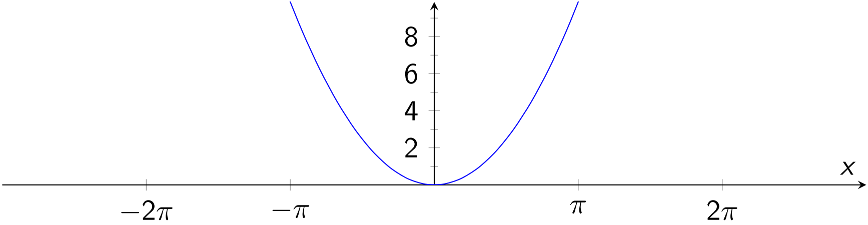


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \underbrace{x^2}_{\text{par}} \underbrace{\text{sen}\, nx}_{\text{impar}} = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, x^2 = \frac{2}{3}\pi^2$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$

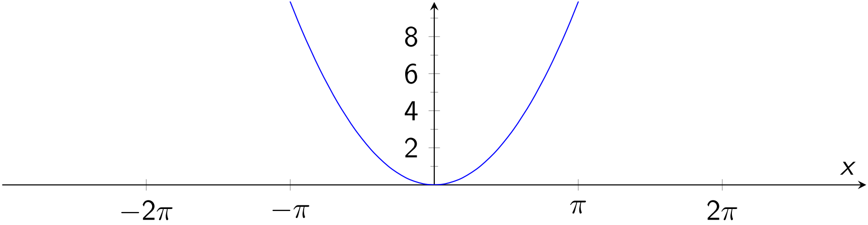


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \underbrace{x^2}_{\text{par}} \underbrace{\text{sen}\, nx}_{\text{impar}} = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 = \frac{2}{3}\pi^2$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$

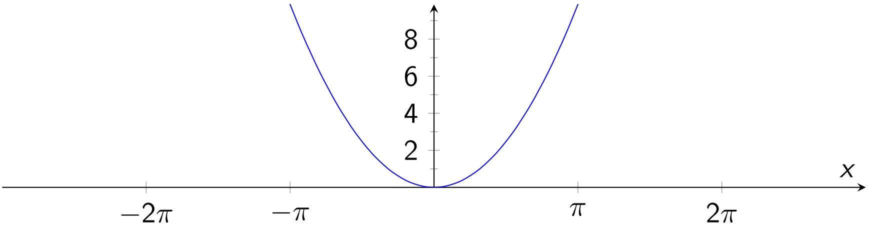


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \; \underbrace{x^2}_{\text{par}} \underbrace{\text{sen}\, nx}_{\text{impar}} = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \; x^2 = \frac{2}{3}\pi^2$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$

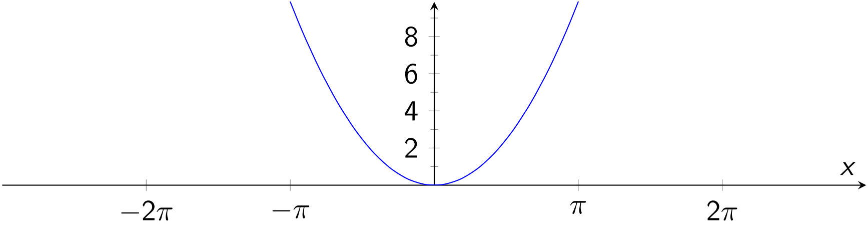


$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx \underbrace{x^2}_{\text{par}} \underbrace{\text{sen}\, nx}_{\text{impar}} = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 = \frac{2}{3}\pi^2$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 e^{inx}$$

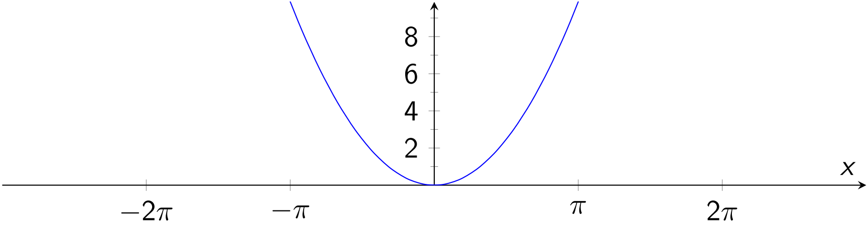
$$= \frac{e^{inx}}{\pi i n} x^2 \Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{in} 2x \qquad \text{(por partes)}$$

$$= -\frac{e^{inx}}{\pi (in)^2} 2x \Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2} 2 \qquad \text{(por partes)}$$

$$= \frac{2}{\pi n^2}\left[\pi e^{in\pi} + \pi e^{-in\pi}\right] + \frac{2}{\pi (in)^3}\left[e^{in\pi} - e^{-in\pi}\right]$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 e^{inx}$$

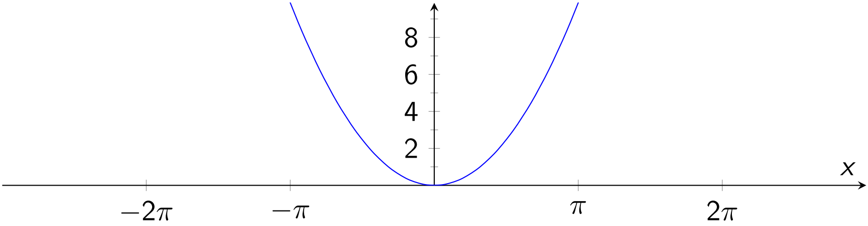
$$= \frac{e^{inx}}{\pi in}x^2\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{in}2x \qquad \text{(por partes)}$$

$$= -\frac{e^{inx}}{\pi(in)^2}2x\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2}2 \qquad \text{(por partes)}$$

$$= \frac{2}{\pi n^2}\left[\pi e^{in\pi} + \pi e^{-in\pi}\right] + \frac{2}{\pi(in)^3}\left[e^{in\pi} - e^{-in\pi}\right]$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\; x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\; x^2 e^{inx}$$

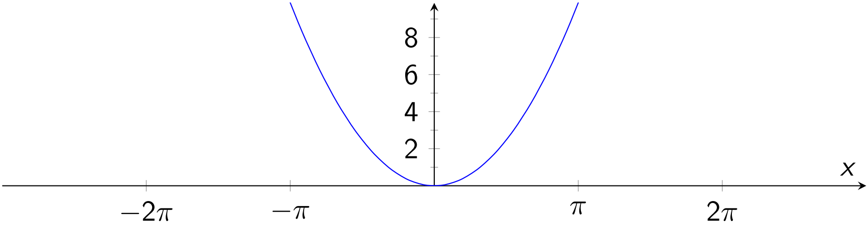
$$= \frac{e^{inx}}{\pi in}x^2\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{in}2x \qquad \text{(por partes)}$$

$$= -\frac{e^{inx}}{\pi(in)^2}2x\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2}2 \qquad \text{(por partes)}$$

$$= \frac{2}{\pi n^2}\left[\pi e^{in\pi} + \pi e^{-in\pi}\right] + \frac{2}{\pi(in)^3}\left[e^{in\pi} - e^{-in\pi}\right]$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 e^{inx}$$

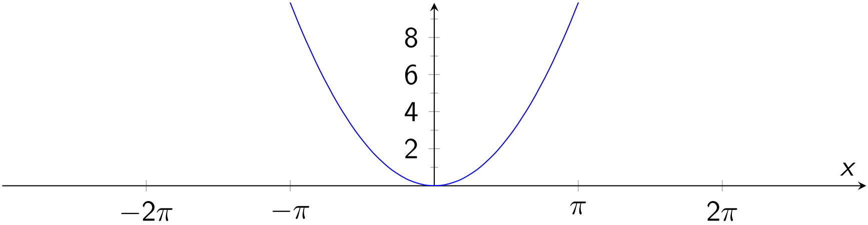
$$= \frac{e^{inx}}{\pi in}x^2\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{in}2x \qquad \text{(por partes)}$$

$$= -\frac{e^{inx}}{\pi(in)^2}2x\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2}2 \qquad \text{(por partes)}$$

$$= \frac{2}{\ldots}\left[\pi e^{in\pi} + \pi e^{-in\pi}\right] + \frac{2}{\ldots}\left[e^{in\pi} - e^{-in\pi}\right]$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



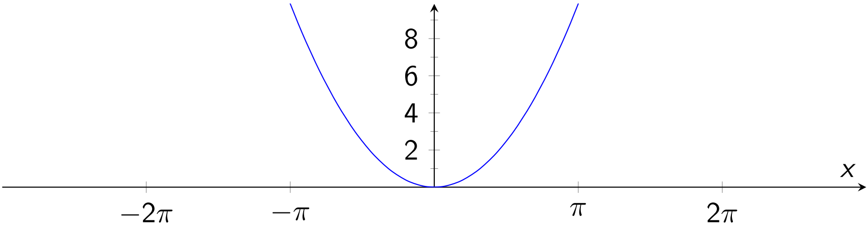
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 e^{inx}$$

$$= \cancel{\frac{e^{inx}}{\pi in} x^2 \Big|_{-\pi}^{\pi}} - \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{in} 2x \qquad \text{(por partes)}$$

$$= -\frac{e^{inx}}{\pi(in)^2} 2x \Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2} 2 \qquad \text{(por partes)}$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



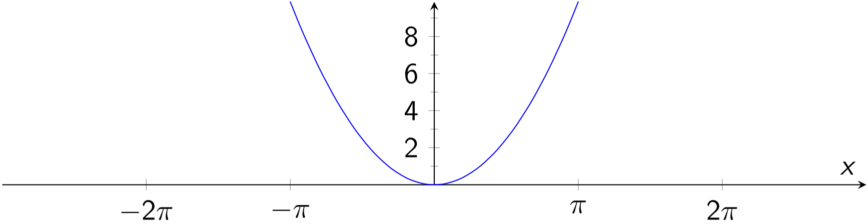
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 e^{inx}$$

$$= \frac{e^{inx}}{\pi in}x^2\Big|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{in}2x \qquad \text{(por partes)}$$

$$= -\frac{e^{inx}}{\pi(in)^2}2x\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2}2 \qquad \text{(por partes)}$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 e^{inx}$$

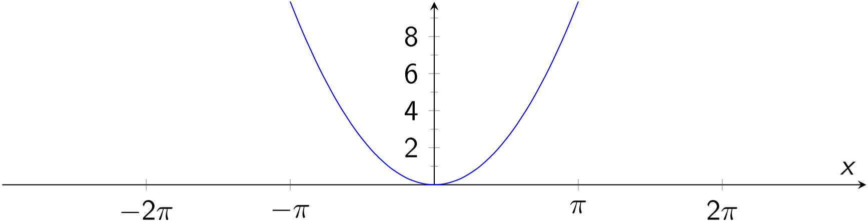
$$= -\frac{e^{inx}}{\pi(in)^2}2x\Big|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2}2 \qquad \text{(por partes)}$$

$$= \frac{2}{\pi n^2}\left[\pi e^{in\pi} + \pi e^{-in\pi}\right] + \frac{2}{\pi(in)^3}\left[e^{in\pi} - e^{-in\pi}\right]$$

$$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

Exemplo 1: parábola

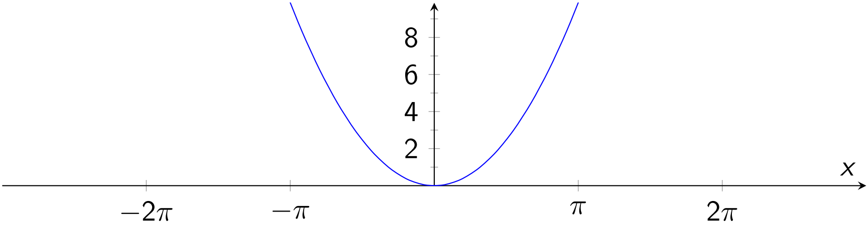
Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 \cos nx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x^2 e^{inx} \\
&= -\frac{e^{inx}}{\pi(in)^2}2x\bigg|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} dx \frac{e^{inx}}{(in)^2}2 \qquad \text{(por partes)} \\
&= \frac{2}{\pi n^2}\left[\pi e^{in\pi} + \pi e^{-in\pi}\right] + \cancel{\frac{2}{\pi(in)^3}\left[e^{in\pi} - e^{-in\pi}\right]} \\
&= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n
\end{aligned}
$$

Exemplo 1: parábola

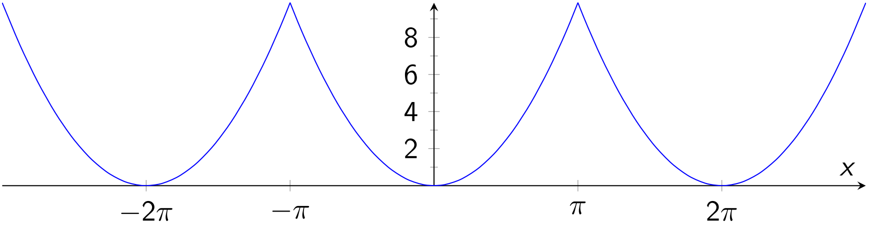
Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Exemplo 1: parábola

Considere a função $f(x) = x^2$ no intervalo $[-\pi, \pi]$



$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

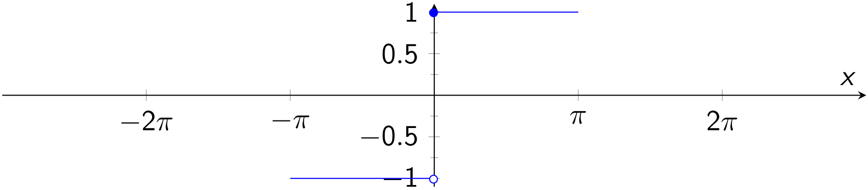
Simetria

Fica claro, através desse exemplo, que se

- $f(x)$ é par $\implies b_n = 0$ (não há termos com sen nx)
- $f(x)$ é impar $\implies a_n = 0$ (não há termos com cos nx)

Exemplo 2: Função sinal

Considere a função *descontínua*, $f(x) = \begin{cases} -1 & \text{se } x < 0 \\ +1 & \text{se } x \geq 0 \end{cases}$



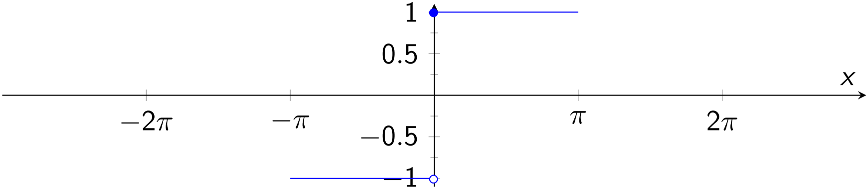
É uma função (quase) *impar*. Logo, esperamos que $a_n = 0$. De fato:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\cos nx) + \frac{1}{\pi}\int_{0}^{\pi} dx \cos nx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} dx \cos nx + \frac{1}{\pi}\int_{0}^{\pi} dx \cos nx = 0$$

Exemplo 2: Função sinal

Considere a função *descontínua*, $f(x) = \begin{cases} -1 & \text{se } x < 0 \\ +1 & \text{se } x \geq 0 \end{cases}$



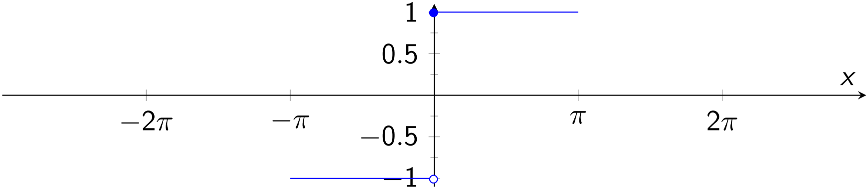
É uma função (quase) *impar*. Logo, esperamos que $a_n = 0$. De fato:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\cos nx) + \frac{1}{\pi}\int_{0}^{\pi} dx \cos nx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} dx \cos nx + \frac{1}{\pi}\int_{0}^{\pi} dx \cos nx = 0$$

Exemplo 2: Função sinal

Considere a função *descontínua*, $f(x) = \begin{cases} -1 & \text{se } x < 0 \\ +1 & \text{se } x \geq 0 \end{cases}$



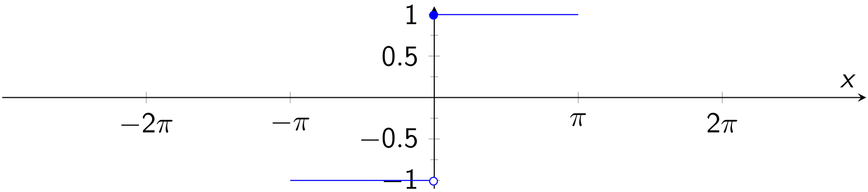
É uma função (quase) *impar*. Logo, esperamos que $a_n = 0$. De fato:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\cos nx) + \frac{1}{\pi}\int_{0}^{\pi} dx \cos nx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} dx \cos nx + \frac{1}{\pi}\int_{0}^{\pi} dx \cos nx = 0$$

Exemplo 2: Função sinal

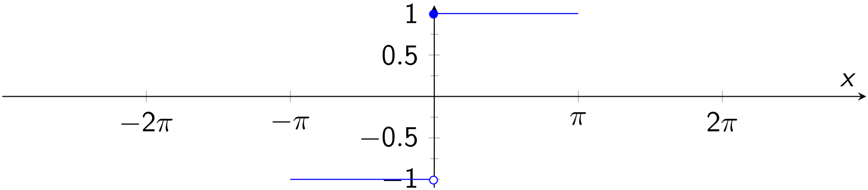
Considere a função *descontínua*, $f(x) = \begin{cases} -1 & \text{se } x < 0 \\ +1 & \text{se } x \geq 0 \end{cases}$



É uma função (quase) *impar*. Logo, esperamos que $a_n = 0$. De fato:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\cos nx) + \frac{1}{\pi}\int_{0}^{\pi} dx \cos nx$$
$$= -\frac{1}{\pi}\int_{0}^{\pi} dx\cos nx + \frac{1}{\pi}\int_{0}^{\pi} dx\cos nx = 0$$

Exemplo 2: Função sinal



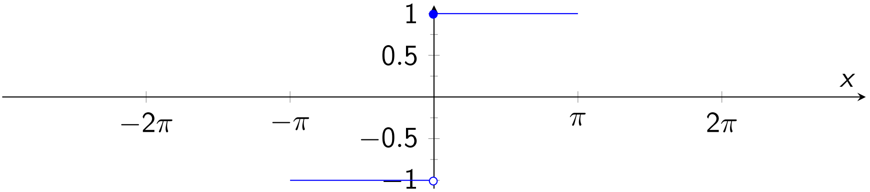
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}\, nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\text{sen}\, nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\, nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\text{sen}\, nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\, nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\text{sen}\, nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_{0}^{\pi} = \frac{2}{\pi n}\left[1-\cos n\pi\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal



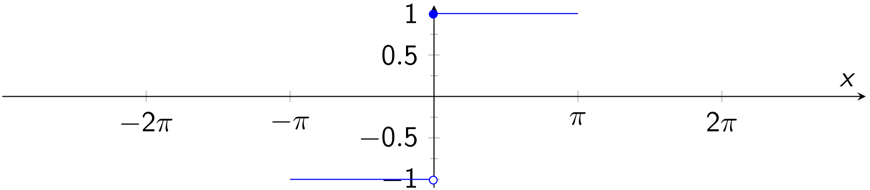
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}\,nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\mathrm{sen}\,nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\mathrm{sen}\,nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\mathrm{sen}\,nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\mathrm{sen}\,nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\mathrm{sen}\,nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_{0}^{\pi} = \frac{2}{\pi n}\left[1-\cos n\pi\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal



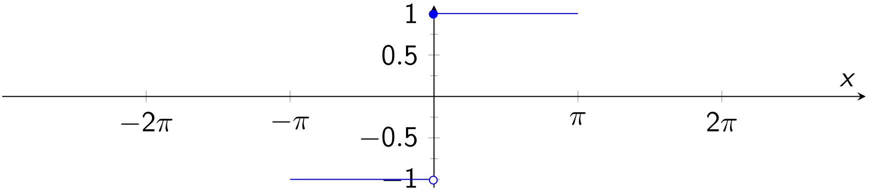
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}\, nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\mathrm{sen}\, nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\mathrm{sen}\, nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\mathrm{sen}\, nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\mathrm{sen}\, nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\mathrm{sen}\, nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{2}{\pi n}\left[1-\cos n\pi\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal



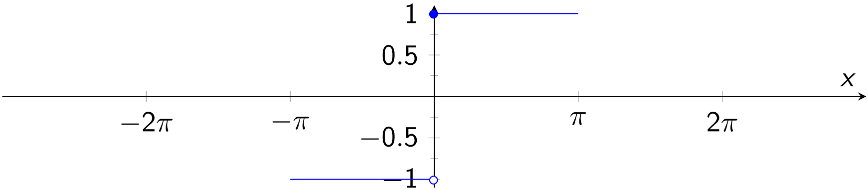
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}\,nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\text{sen}\,nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\text{sen}\,nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_{0}^{\pi} = \frac{2}{\pi n}\left[1-\cos n\pi\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal



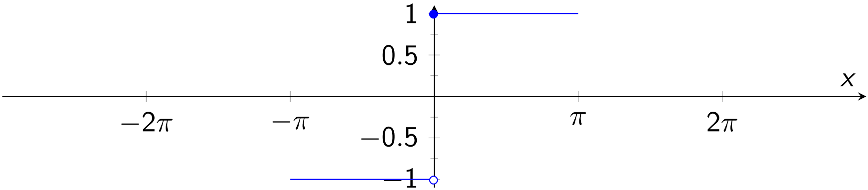
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}\,nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\text{sen}\,nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\text{sen}\,nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\bigg|_{0}^{\pi} = \frac{2}{\pi n}[1-\cos n\pi] = \frac{2}{\pi n}[1-(-1)^n]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal



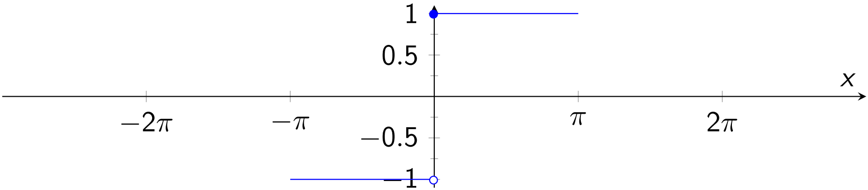
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}\,nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\text{sen}\,nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\text{sen}\,nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_{0}^{\pi} = \frac{2}{\pi n}\left[1-\cos n\pi\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal



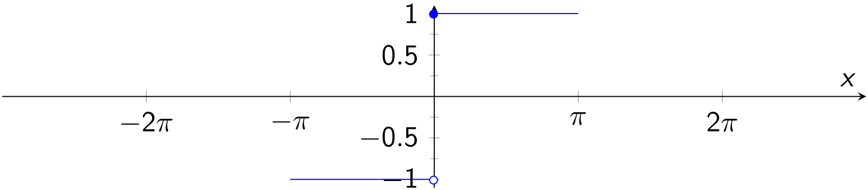
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}\,nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\text{sen}\,nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\text{sen}\,nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\text{sen}\,nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_{0}^{\pi} = \frac{2}{\pi n}\left[1-\cos n\pi\right] = \frac{2}{\pi n}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal



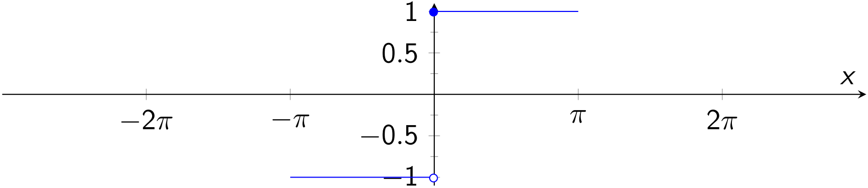
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}\, nx = \frac{1}{\pi}\int_{-\pi}^{0} dx(-\,\text{sen}\, nx) + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\, nx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \underset{(x\to -x)}{dx\,\text{sen}\, nx} + \frac{1}{\pi}\int_{0}^{\pi} dx\,\text{sen}\, nx = \frac{2}{\pi}\int_{0}^{\pi} dx\,\text{sen}\, nx$$

$$= -\frac{2}{\pi}\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{2}{\pi n}\left[1 - \cos n\pi\right] = \frac{2}{\pi n}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 & \text{se } n = \text{ par} \\ \frac{4}{\pi n} & \text{se } n = \text{ impar} \end{cases}$$

Exemplo 2: Função sinal

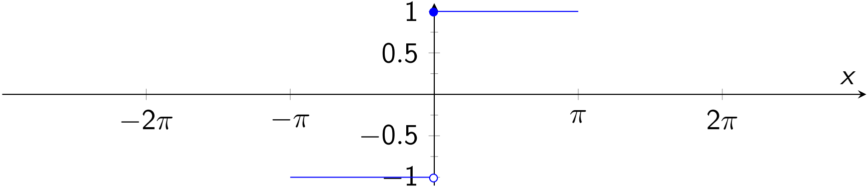


$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \operatorname{sen} nx$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\operatorname{sen}[(2n+1)x]}{2n+1}$$

$f(0) = 0, \rightarrow$ média aritmética de $\lim_{x\to 0^-} f(x) = -1$ e $\lim_{x\to 0^+} f(x) = 1$

Exemplo 2: Função sinal

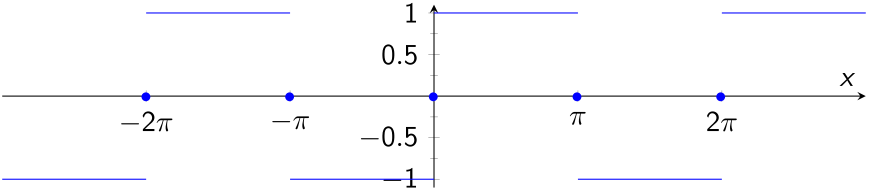


$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \,\mathrm{sen}\, nx$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\mathrm{sen}[(2n+1)x]}{2n+1}$$

$f(0) = 0, \rightarrow$ média aritmética de $\lim_{x\to 0^-} f(x) = -1$ e $\lim_{x\to 0^+} f(x) = 1$

Exemplo 2: Função sinal

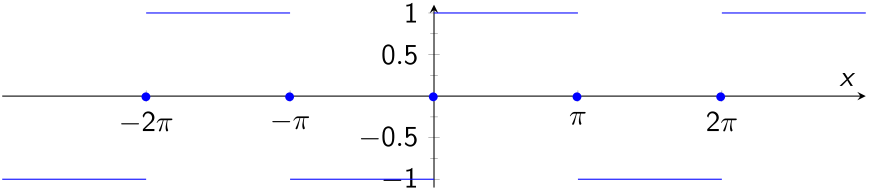


$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \operatorname{sen} nx$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\operatorname{sen}[(2n+1)x]}{2n+1}$$

$f(0) = 0, \rightarrow$ média aritmética de $\lim_{x\to 0^-} f(x) = -1$ e $\lim_{x\to 0^+} f(x) = 1$

Exemplo 2: Função sinal



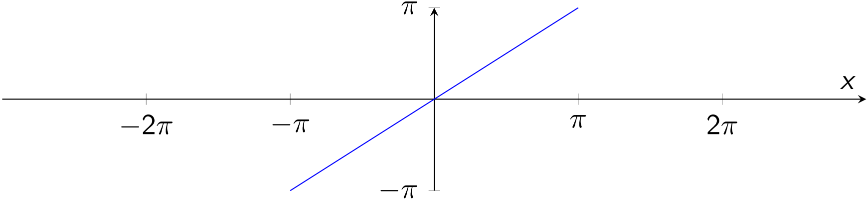
$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \operatorname{sen} nx$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\operatorname{sen}[(2n+1)x]}{2n+1}$$

$f(0) = 0, \rightarrow$ média aritmética de $\lim_{x\to 0^-} f(x) = -1$ e $\lim_{x\to 0^+} f(x) = 1$

Exemplo 3: Onda de serra

Considere a função $f(x) = x \qquad \text{se} \;\; -\pi \leq x \leq \pi$



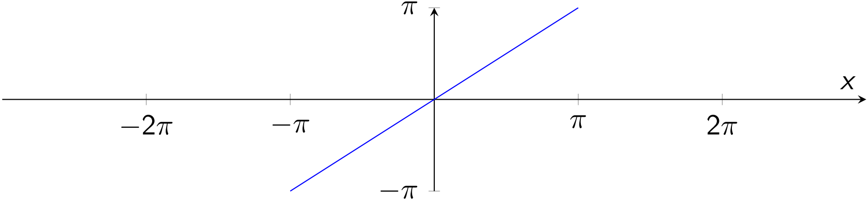
De novo é uma função *impar* $\implies a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, x \,\text{sen}\, nx = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx \frac{\cos nx}{n}$$

$$= \frac{2(-1)^{n+1}}{n} + \frac{1}{\pi}\frac{\text{sen}\, nx}{n}\Big|_{-\pi}^{\pi} \to 0$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\,\text{sen}\, nx$$

Considere a função $f(x) = x \quad \text{se} \quad -\pi \leq x \leq \pi$



De novo é uma função *impar* $\implies a_n = 0$.

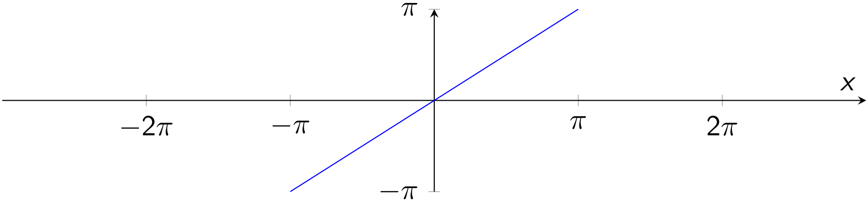
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, x \operatorname{sen} nx = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx \frac{\cos nx}{n}$$

$$= \frac{2(-1)^{n+1}}{n} + \frac{1}{\pi}\frac{\operatorname{sen} nx}{n}\Big|_{-\pi}^{\pi} \to 0$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$

Exemplo 3: Onda de serra

Considere a função $f(x) = x \qquad \text{se} \quad -\pi \leq x \leq \pi$



De novo é uma função *impar* $\implies a_n = 0$.

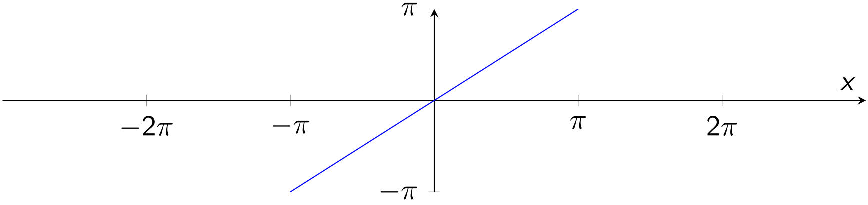
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x \,\text{sen}\, nx = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx \frac{\cos nx}{n}$$

$$= \frac{2(-1)^{n+1}}{n} + \frac{1}{\pi}\frac{\text{sen}\, nx}{n}\Big|_{-\pi}^{\pi} \rightarrow 0$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\,\text{sen}\, nx$$

Exemplo 3: Onda de serra

Considere a função $f(x) = x \qquad \text{se} \quad -\pi \leq x \leq \pi$



De novo é uma função *impar* $\implies$ $a_n = 0$.

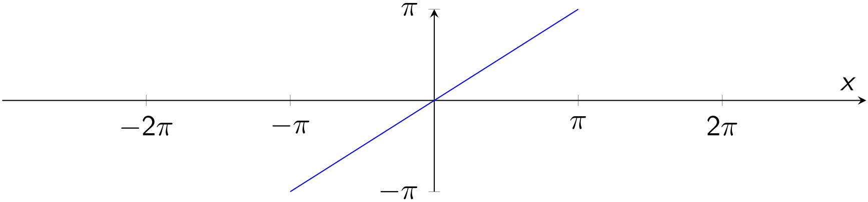
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x \operatorname{sen} nx = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx \frac{\cos nx}{n}$$

$$= \frac{2(-1)^{n+1}}{n} + \cancelto{0}{\frac{1}{\pi}\frac{\operatorname{sen} nx}{n}\Big|_{-\pi}^{\pi}}$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$

Exemplo 3: Onda de serra

Considere a função $f(x) = x \qquad \text{se} \quad -\pi \leq x \leq \pi$



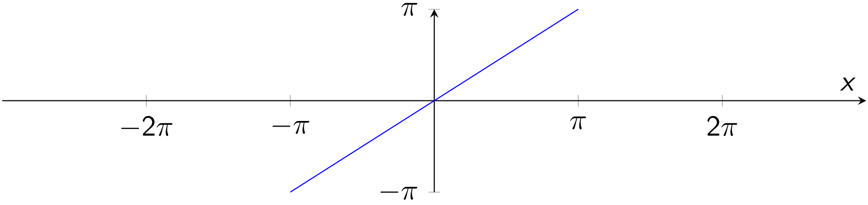
De novo é uma função *impar* $\implies a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x \,\text{sen}\, nx = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx \frac{\cos nx}{n}$$

$$= \frac{2(-1)^{n+1}}{n} + \frac{1}{\pi}\frac{\text{sen}\, nx}{n}\Big|_{-\pi}^{\pi} \rightarrow 0$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \,\text{sen}\, nx$$

Considere a função $f(x) = x \qquad \text{se} \quad -\pi \leq x \leq \pi$



De novo é uma função *impar* $\implies a_n = 0$.

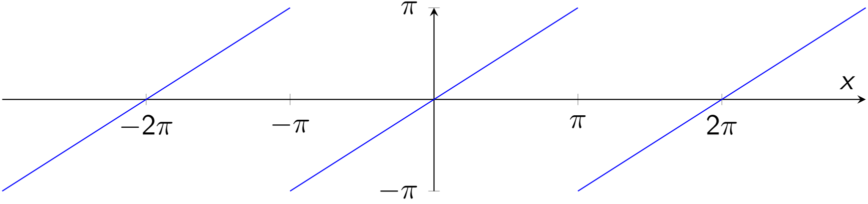
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x \,\text{sen}\, nx = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx \frac{\cos nx}{n}$$

$$= \frac{2(-1)^{n+1}}{n} + \cancelto{0}{\frac{1}{\pi}\frac{\text{sen}\, nx}{n}\Big|_{-\pi}^{\pi}}$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\,\text{sen}\, nx$$

Exemplo 3: Onda de serra

Considere a função $f(x) = x \qquad \text{se} \quad -\pi \leq x \leq \pi$



De novo é uma função *impar* $\implies a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx x \,\text{sen}\, nx = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} dx \frac{\cos nx}{n}$$

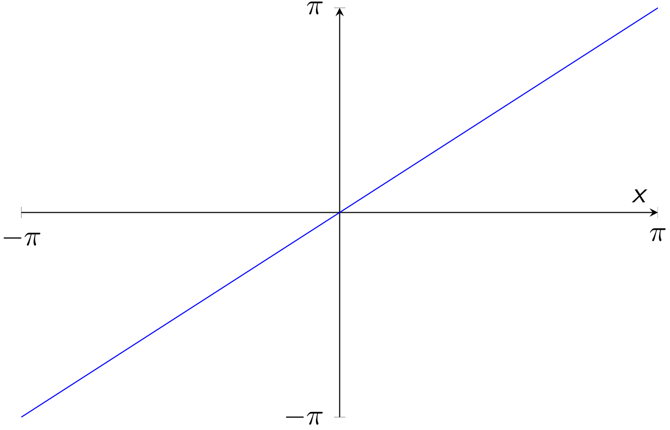
$$= \frac{2(-1)^{n+1}}{n} + \cancelto{0}{\frac{1}{\pi}\frac{\text{sen}\, nx}{n}\bigg|_{-\pi}^{\pi}}$$

$$\implies f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \,\text{sen}\, nx$$

Exemplo 3: Onda de serra

Imagine agora que consideremos apenas um número finito de termos:

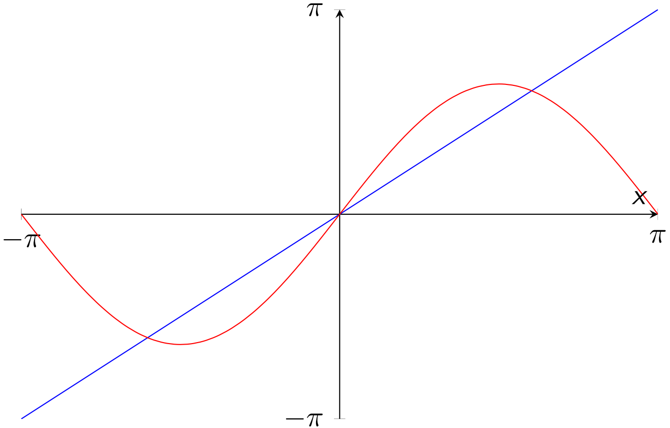
$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$



Exemplo 3: Onda de serra

Imagine agora que consideremos apenas um número finito de termos:

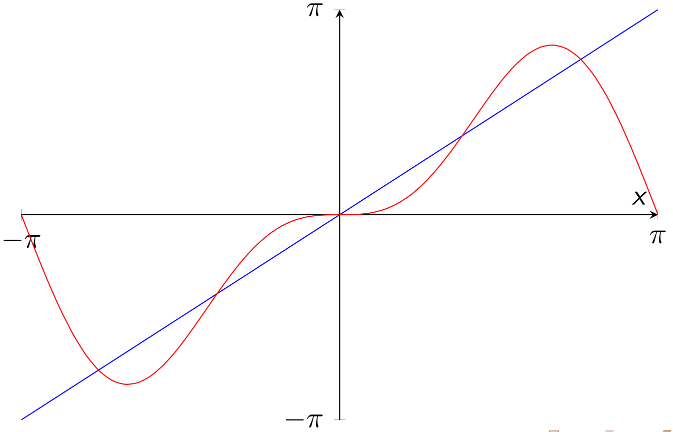
$$f(x) = \sum_{n=1}^{1} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$



Exemplo 3: Onda de serra

Imagine agora que consideremos apenas um número finito de termos:

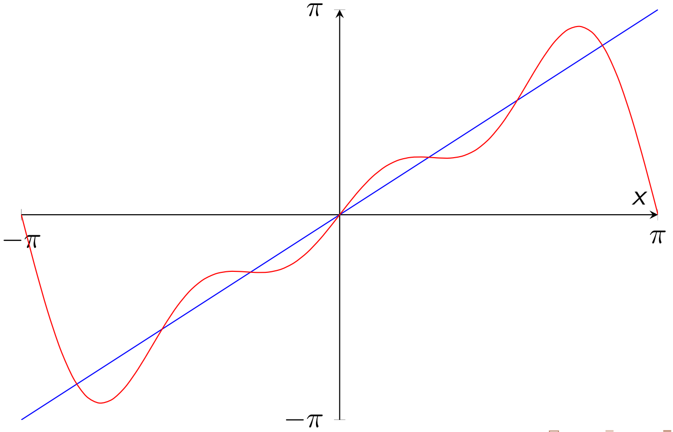
$$f(x) = \sum_{n=1}^{2} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$



Exemplo 3: Onda de serra

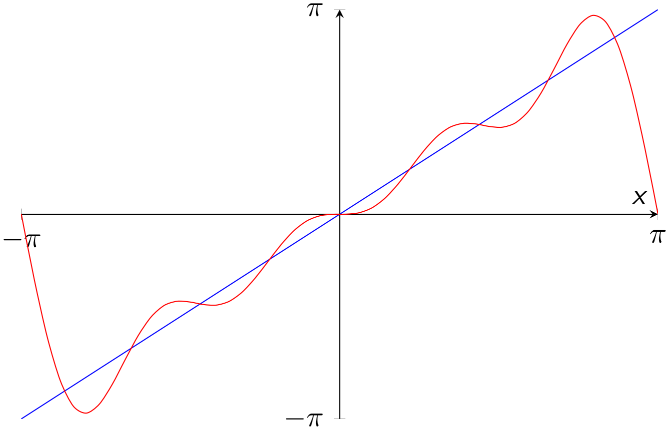
Imagine agora que consideremos apenas um número finito de termos:

$$f(x) = \sum_{n=1}^{3} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$



Imagine agora que consideremos apenas um número finito de termos:

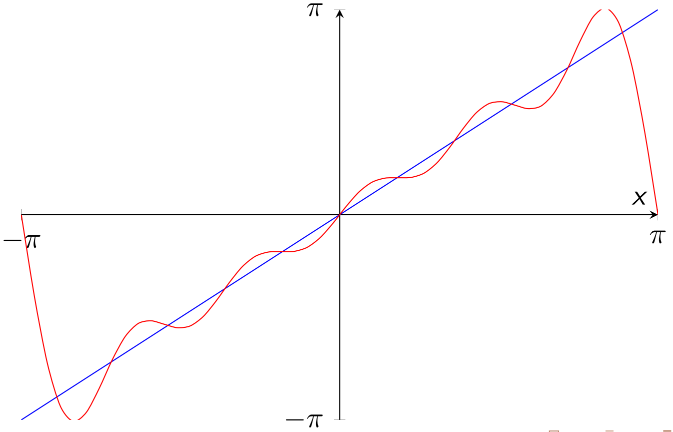
$$f(x) = \sum_{n=1}^{4} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$



Exemplo 3: Onda de serra

Imagine agora que consideremos apenas um número finito de termos:

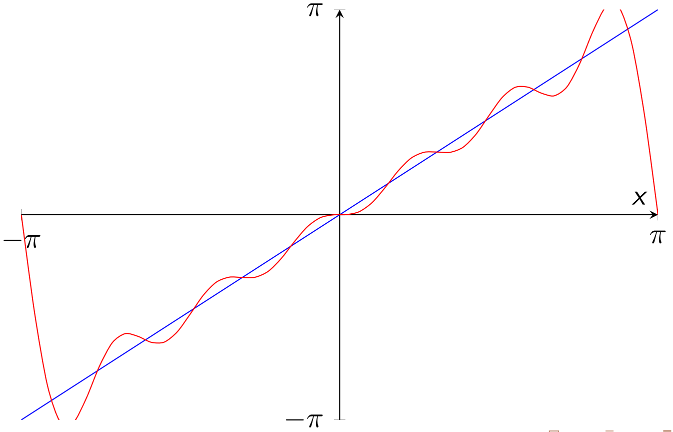
$$f(x) = \sum_{n=1}^{5} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$



Exemplo 3: Onda de serra

Imagine agora que consideremos apenas um número finito de termos:

$$f(x) = \sum_{n=1}^{6} \frac{2(-1)^{n+1}}{n} \operatorname{sen} nx$$

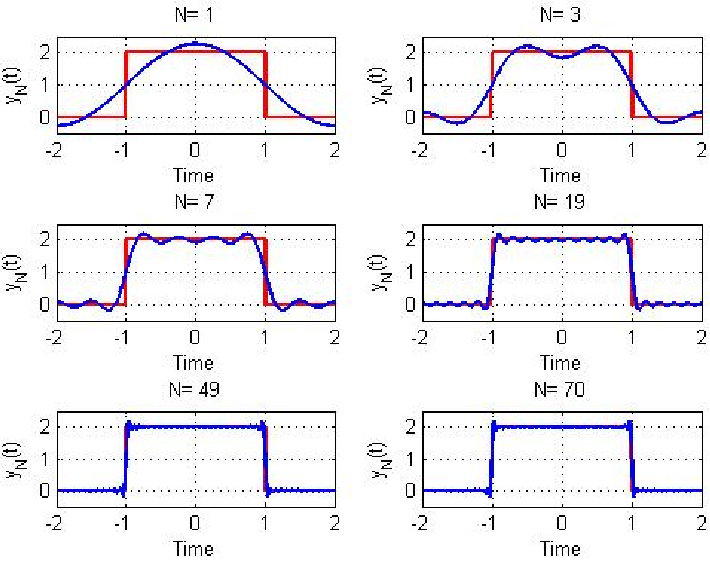


Fenômeno do Gibbs

Note que:

1. Quando incluímos mais termos na série a acurácia com a qual a série finita representa $f(x)$ aumenta.
2. Todas as curvas vão para zero quando $x = \pi$.
3. Na vizinhança de $x = \pi$ (onde a descontinuidade aparece) a série tem um "overshoot" (ultrapassagem) que persiste e não diminui quando incluímos mais termos na série. Esse é um exemplo do conhecido "fenômeno do Gibbs", que vocês podem ler mais sobre no livro de Arfken (cap. 14).

Fenômeno do Gibbs



Mudança de intervalo

- Pode construir um série de Fourier para descrever uma função em qualquer intervalo finito (qualquer período).
- Para período $2L$, podemos escrever

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \operatorname{sen}\left(\frac{n\pi x}{L}\right) \right]$$

- (Note que $\cos\left(\frac{n\pi x}{L}\right)$ tem período $2L$.)
- Os coeficientes ficarão

$$a_n = \frac{1}{L} \int_{-L}^{L} dx \; f(x) \cos\left(\frac{n\pi x}{L}\right), \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} dx \; f(x) \operatorname{sen}\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, \ldots$$

- Qualquer intervalo $(x_0, x_0 + 2L)$ serve.

Série de Fourier em senos *ou* co-senos

Vamos definir coeficientes calculados com integrais sobre a metade desse domínio:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

e formar a série com somente co-senos:

$$g_c(x) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

Esta série representa a extensão par (ou simétrica) de $f(x)$:

$$g_c(x) = \begin{cases} f(x), & 0 < x < L \\ f(-x), & -L < x < 0 \end{cases}$$

Série de Fourier em senos *ou* co-senos

Vamos definir coeficientes calculados com integrais sobre a metade desse domínio:

$$b_n = \frac{2}{L}\int_0^L f(x)\,\mathrm{sen}\left(\frac{n\pi x}{L}\right) dx$$

e formar a série com somente senos:

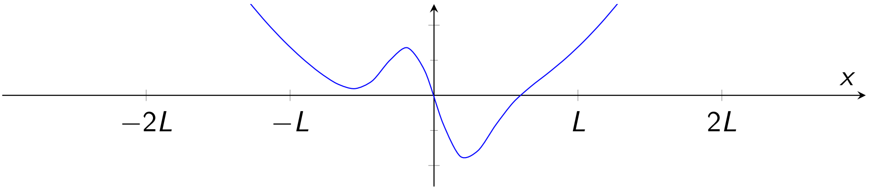
$$g_s(x) \equiv \sum_{n=1}^{\infty} b_n\,\mathrm{sen}\left(\frac{n\pi x}{L}\right),$$

Esta série representa a extensão impar (ou anti-simétrica) de $f(x)$:

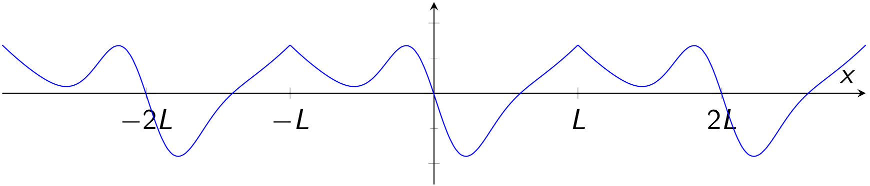
$$g_s(x) = \begin{cases} f(x), & 0 < x < L \\ -f(-x), & -L < x < 0 \end{cases}$$

Série de Fourier em senos *ou* co-senos

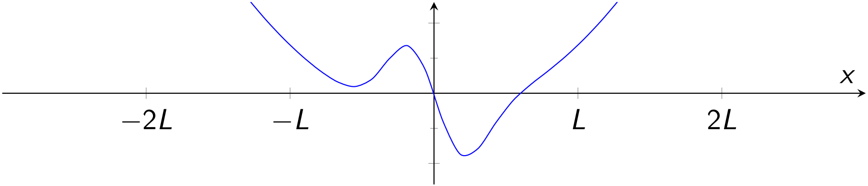
Por exemplo, suponha que $f(x)$ seja



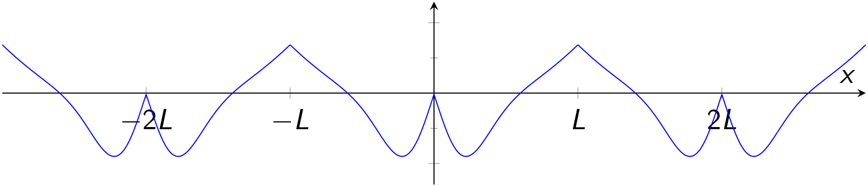
A série de Fourier (com período $2L$) será



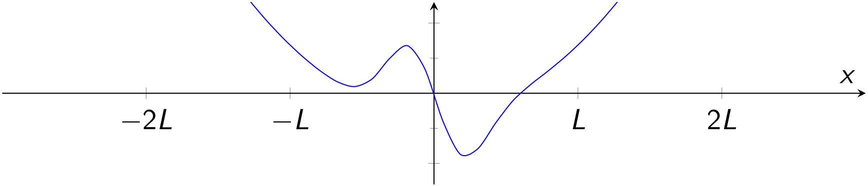
Por exemplo, suponha que $f(x)$ seja



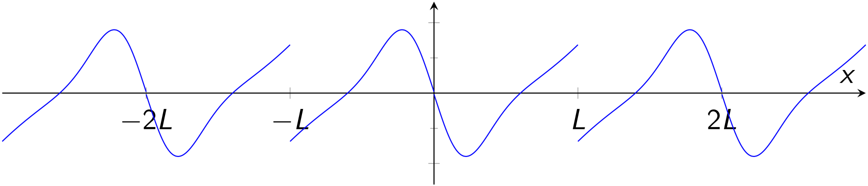
A série em co-senos será



Por exemplo, suponha que $f(x)$ seja



A série em senos será



Fatos interessantes

1. Naquele exemplo onde

$$f(x) = \begin{cases} -1 & \text{se } x < 0 \\ +1 & \text{se } x \geq 0 \end{cases} \tag{1}$$

a série de Fourier era tal que o n-ésimo coeficiente ia $\sim \frac{1}{n}$. Esse tipo de dependência é o meso que ocorre na série harmônica, que é divergente. Convergência, nesse caso para esse tipo de série é bem lenta. O fato de que $a_n \sim \frac{1}{n}$ geralmente ocorre quando a função tem descontinuidades.

2. Se $f(x)$ é contínua (mas talvez com derivadas descontínuas), geralmente o n-ésimo coeficiente $\sim \frac{1}{n^2}$ (convergência é muito mais rápida).